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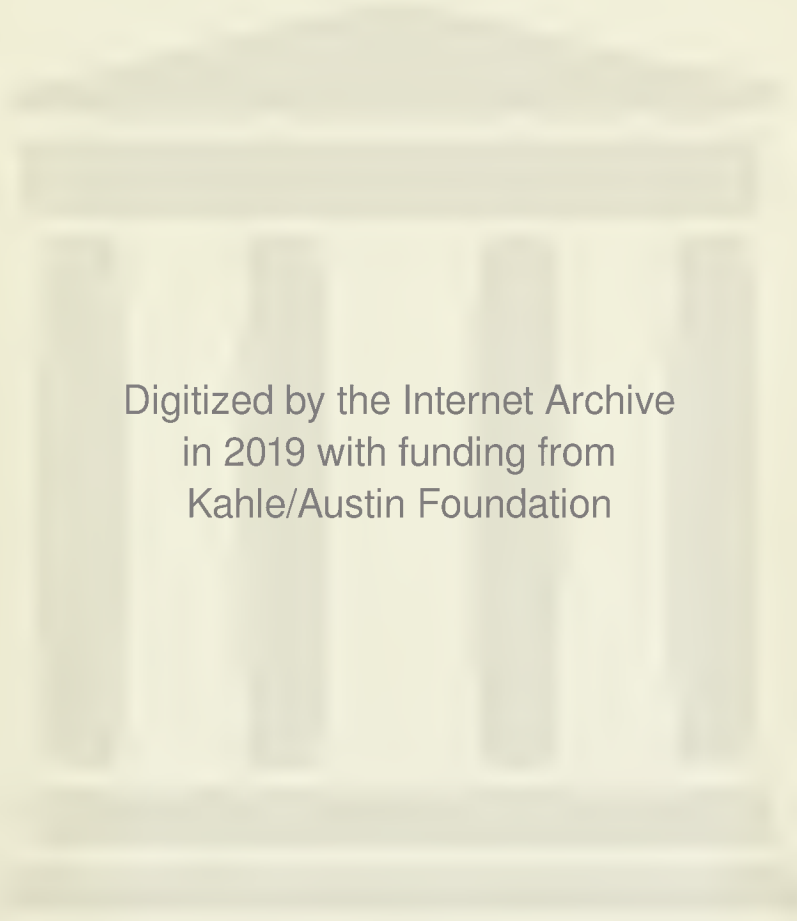

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WHOLES AND PARTS

A GENERAL THEORY OF SYSTEM BEHAVIOUR

WHOLES AND PARTS

A GENERAL THEORY OF SYSTEM BEHAVIOUR

by

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I. INTRODUCTION

One of the key questions of dialectical materialism is that of the whole and of the nature of dialectical processes of development. Dialectical materialism asserts the existence of material systems, the elements of which are linked by a chain of cause-and-effect relations. Such systems possess attributes distinct from those of their constituent elements and they also have their own modes of action which cannot be derived only from those of the elements. We call such systems "wholes". Instances of wholes of this sort are chemical compounds as compared with physical aggregates of elements, living matter as compared with physico-chemical systems, animal organisms in comparison with individual cells of living matter, psychical processes in comparison with the biological properties of an organism and, finally, human societies or to be more precise, various historical social formations, in comparison with the biopsychical properties and modes of behaviour of the human organism.

According to the theory of dialectical materialism, the existence of such wholes is connected with the dialectical nature of development processes. In a system constituting a whole there appear contradictions which prevent the system from remaining in a changeless state. The contradictions in the system induce changes leading to a readjustment which makes the contradictions disappear. But these very changes open the way to new contradictions which, in turn, induce new changes-etc. As a result, wholes can never remain in a changeless state; they must change constantly. The changes, however, show a definite direction; in other words: they represent a *process of development.* In the course of development, individual wholes combine into more complex systems, into wholes "of higher order" which exhibit new properties and new modes of action hitherto not

encountered. Thus, in the course of dialectical development *new properties* (new "qualities") and new modes of action (new laws of behaviour) come into being.

This picture of the relation of wholes to their elements and of the manner in which, in the course of dialectical development, new wholes appear with their corresponding modes of action is meant to be a broad generalization of the results of natural and of social science. The advancement of science confirms it increasingly. Scientific progress also makes possible to deepen the methodological structure of this picture. And there is need of it. Attempts at a more precise analysis of the picture presented, which increasingly is borne out by experience, readily lead to one-sided conceptions which do not conform to experience nor to scientific methodology.

One type of such conceptions is *metaphysical finalism*. It imputes to wholes some specific substantial factor which allegedly imparts to them the nature of a whole with distinct properties and modes of action. This factor ensures the directional nature of development. In the form of theories about "vital force", "entelechy", "élan vital", "soul", "spirit of a nation or epoch", etc., metaphysical finalism has appeared, and still appears, in philosophy as well as in the various sciences. It claims justification in the fact that new properties and modes of action appear at "higher" levels of development. As we know, some representatives of dialectical materialism, strongly influenced by Hegel, have been affected by conception of this kind.

The second type of one-sided conception is *mechanicism*. It consists in the view that the properties and modes of action of a whole are entirely due to the properties of its constituent elements. The denial of this proposition is considered as metaphysical finalism. Some representatives of dialectical materialism were close to such a mechanistic approach to the subject. In taking their view, they were guided by the fear of falling into a metaphysical way of thinking.

Both these conceptions, however, are in conflict with experience and with scientific method. Mechanicism denies the experiential fact of the existence of wholes which have their

own specific properties and modes of action. Finalism, on the other hand, introduces "entities" the existence of which is not established by experience, and never can be confirmed. A precise and methodologically correct approach to the question of wholes and of dialectical development was, so far prevented by the lack of an intellectual apparatus—concepts and rules of operating with them—adequate for the purpose. At present such an apparatus is beginning to emerge in connection with the development of a new science, i.e. cybernetics. The present essay tries to present the outlines of the intellectual apparatus required for precise treatment of the problem of the whole and of dialectical development.

II. THE ACTIVE ELEMENT

By an *active element* E we mean a material object which depends in a given manner on other material objects and acts in a given manner on other material objects. The set of other material objects shall be called the *environment* of the given element.

We assume that:

1. The environment acts on element E by inducing in it certain states of a strictly defined type such as temperature, pressure, electric charge, feeling, sense impression. The individual types of such states shall be called the *inputs* of element E.

2. Element E acts on the environment by assuming certain states of a strictly defined character, e.g. temperature, magnetic field, colour, generation of sounds, motion. The individual types of such states shall be called the *outputs* of element E.

3. Element E possesses at least one input and at least one output.

4. The input states determine the output states *uniquely*.

Assumptions 1 and 2 are also expressed by the statement that element E is "relatively isolated".[1] This means that the element's contact with the environment takes place solely through the intermediary of its inputs and outputs. There are no other "channels" of contact. Assumption 3 excludes elements without inputs and outputs, i.e. *monads*. It also excludes elements with

[1] This term was first introduced by Henryk Greniewski in *Elementy logiki indukcji* (*Elements of the Logic of Induction*, in Polish), PWN, Warsaw, 1955, p. 28. Cf. also Greniewski, *Cybernetics Without Mathematics*, Pergamon Press, London, PWN, Warsaw, 1960, pp. 9–10. Greniewski speaks of a "relatively isolated system". In view of the subject of the present paper, however, we prefer the term "active element" leaving the term "system" to denote a set of coupled elements. Cf. page 17.

only inputs or only outputs, in other words, solely passive and solely active elements, so to speak. Each element through its inputs "receives" the action of the environment and through its outputs, "transmits" action into the environment.

Assumption 4 states the unique relation between the input and output states. We shall call this relation the *mode of action* of element E.

For the time being, let us assume that the inputs and outputs of element E are finite in number. Let us denote the number of inputs by m and the number of outputs by n. Both m and n are natural numbers. In view of assumption 3, $m \geqslant 1$ and $n \geqslant 1$. The difference

$$\delta = n - m \tag{2.1}$$

will be called the *diversification* of element E. Obviously, δ is an integer. It may happen, however, that $\delta > 0$ as well as $\delta < 0$ or $\delta = 0$.

The input and output states of element E can be represented by numbers. If the input or output state consists of certain attributes (e.g. red colour, a given sound or movement) appearing or not, then we denote the state by 1 when the attribute appears and 0 when it does not. If the input or output state has the character of a discretely varying magnitude (e.g. number of flies in an experimental jar, the fraction of male births among the births occuring in a family), we denote the state by a rational number that serves as a measure of that magnitude. If, finally, the input or output state is a magnitude varying continuously (e.g. temperature, weight, height), we denote this state by a real number.

In this way, numbers—let us denote them by $x_1, x_2, \ldots, x_m$—can be assigned to the inputs to express their individual states. These numbers form a vector

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_m), \tag{2.2}$$

which represents the state of inputs. We call it the *input vector* and the numbers $x_1, x_2, \ldots, x_m$ constitute its *components*.

In similar fashion, the states of individual outputs are assigned numbers which we denote by $y_1, y_2, \ldots, y_n$. These are the components of the *output vector*

$$y = (y_1, y_2, \ldots, y_n). \tag{2.3}$$

Both vectors, the input and output, are subject to the rules of vector algebra.

The active element E is represented graphically as follows:

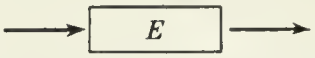

FIG. 1

In this drawing, on the left the input vector is indicated by an arrow and the output vector is on the right, also in the form of an arrow. Sometimes it is useful to show all the components of both vectors on the drawing. To avoid confusion we denote the components with broken lines. Then the drawing assumes the form below

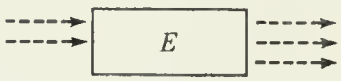

FIG. 2

In Fig. 2 the input vector has two components and the output vector has three.

The relation between the input and output states (in other words, the mode of action of the element E) can be expressed mathematically as a *transformation* of the vector $\boldsymbol{x}$ into the vector $\boldsymbol{y}$. Symbolically, we write this transformation

$$\boldsymbol{y} = T(\boldsymbol{x}). \tag{2.4}$$

The set of admissible values of the vector $\boldsymbol{x}$ is called the *domain* of the transformation; the set of admissible values of the vector $\boldsymbol{y}$ is called the *field* of the transformation. The symbol T is called the *transformation operator*; it expresses the rule according to which the vector $\boldsymbol{x}$ is transformed into the vector $\boldsymbol{y}$.

This rule can be expressed by means of a certain matrix. For this purpose, we denote by Δx_j the change in the value of the jth component of the vector $\boldsymbol{x}$ and by Δy_i the change in the value of the ith component of the vector $\boldsymbol{y}$. Let us assume that the

jth component of the vector $\boldsymbol{x}$ changes by Δx_j and the other components of this vector remain unchanged. Then, in accordance with transformation (2.4) there is a specified change Δy_i in the ith component of the vector $\boldsymbol{y}$. The ratio

$$a_{ij} = \left(\frac{\Delta y_i}{\Delta x_j}\right)_{\Delta x_k = 0 \text{ for } k \neq j} \qquad \begin{pmatrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m \end{pmatrix} \tag{2.5}$$

is a measure of the partial effect of a change in the jth component of the vector $\boldsymbol{x}$ on the ith component of the vector $\boldsymbol{y}$. We call it the *coefficient of partial effect.*

The coefficients of partial effect form a matrix of n rows and m columns which we shall denote by A. We then have

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{bmatrix}. \tag{2.6}$$

We call this matrix the *transformation matrix.*

With the aid of the transformation matrix the rule for the transformation of the vector $\boldsymbol{x}$ into the vector $\boldsymbol{y}$ can be expressed in the form of a system of equations.

Namely,

$$\begin{aligned} \Delta y_1 &= a_{11}\Delta x_1 + a_{12}\Delta x_2 + \ldots + a_{1m}\Delta x_m, \\ \Delta y_2 &= a_{21}\Delta x_1 + a_{22}\Delta x_2 + \ldots + a_{2m}\Delta x_m, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta y_n &= a_{n1}\Delta x_1 + a_{n2}\Delta x_2 + \ldots + a_{nm}\Delta x_m. \end{aligned} \tag{2.7}$$

This can be expressed even more simply in the form of a vector equation

$$\Delta \boldsymbol{y} = A\Delta \boldsymbol{x} \tag{2.8}$$

where $\Delta \boldsymbol{x}$ and $\Delta \boldsymbol{y}$ are vectors, viz. $\Delta \boldsymbol{x} = (\Delta x_1, \Delta x_2, \ldots, \Delta x_m)$ and $\Delta \boldsymbol{y} = (\Delta y_1, \Delta y_2, \ldots, \Delta y_n)$.

As can be seen, the transformation rule, which is symbolized by the operator T, is here represented explicitly in the form of

the algebraic operations constituting the transformation in question.

The expression of the transformation rule in the form of equation (2.7) or (2.8) shall henceforth be called the *difference form* of this rule. If the coefficients of the partial effect a_{ij} are constant, we immediately obtain the *integral form* of the transformation rule. Namely,

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m, \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m \end{aligned} \tag{2.9}$$

or, more simply,

$$y = Ax. \tag{2.10}$$

Arbitrary constants may be added on the right-hand side of (2.9) and (2.10). For the sake of simplicity we take them to be equal to zero.

The difference form shows the relation between the changes in the vector x and the changes in the vector y. The integral form shows the relation between the values of the vectors x and y. When the coefficients a_{ij} are constant and the integral form of the transformation, therefore, has the form of equation (2.9) or (2.10) we say that the transformation is *linear*.

If the coefficients a_{ij} are not constant, they are functions of the vector x. The transformation matrix A is then a functional matrix. If the vector x is a continuous variable, then (2.7) or (2.8) are differential equations. The solution of these differential equations, if there is one, consists of a system of functions

$$\begin{aligned} y_1 &= f_1(x_1, x_2, \ldots, x_m), \\ y_2 &= f_2(x_1, x_2, \ldots, x_m), \\ &\cdots\cdots\cdots\cdots \\ y_n &= f_n(x_1, x_2, \ldots, x_m) \end{aligned} \tag{2.11}$$

such that

$$a_{ij} = \frac{\partial f_i}{\partial x_j} \qquad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m). \tag{2.12}$$

This system can be written more simply in the form of a vector function:

$$\boldsymbol{y} = f(\boldsymbol{x}). \tag{2.13}$$

The functions (2.11) or (2.13) constitute the integral form of the rule for the transformation of the vector $\boldsymbol{x}$ into the vector $\boldsymbol{y}$.

The result (2.11) or (2.13) is also obtained in the case when the vector $\boldsymbol{x}$ varies in a discrete way. Equations (2.7) or (2.8) are then difference equations and the functions (2.11) or (2.13) are the solutions of those equations.

The transformation (2.4) of the vector $\boldsymbol{x}$ into the vector y permits the state of the outputs of element E to be determined from the state of the inputs. Sometimes it is possible to make the *inverse transformation*

$$\boldsymbol{x} = T^{-1}(\boldsymbol{y}). \tag{2.14}$$

The inverse transformation permits conclusions as to the state of the inputs of element E to be drawn from the state of its outputs. An instance of such inverse transformation is a medical diagnosis of the causes of an ailment on the basis of its symptoms, or else the deciphering of a coded message.

The inverse transformation reduces to the solution of equations (2.9) or (2.11) or, which amounts to the same thing, of equations (2.10) or (2.13), with respect to the vector $\boldsymbol{x}$, given the value of the vector $\boldsymbol{y}$. As we know from algebra, the possibility of inverse transformation depends on two conditions. Firstly, we must have $n \geqslant m$, for otherwise there would be fewer equations than unknowns (the components of vector $\boldsymbol{x}$ are unknowns and there are m of them). In other words, the diversification of the element E cannot be negative since in that case there would not be enough outputs to draw conclusions as to the state of the inputs. Secondly, the rank of the transformation matrix must be m. In other words, exactly m partial effects of changes in the state of the individual inputs must be linearly independent.

If the inverse transformation is possible, the transformation rule can be written

$$\boldsymbol{x} = A^{-1}\boldsymbol{y} \tag{2.15}$$

for linear transformation, and

$$\boldsymbol{x} = f^{-1}(\boldsymbol{y}) \tag{2.16}$$

for the general case. Here, A^{-1} is the inverse of the matrix A and $f^{-1}(\boldsymbol{y})$ is the inverse vector function of the vector function $f(\boldsymbol{x})$.

III. COUPLING OF ACTIVE ELEMENTS

In accordance with assumptions 1 and 2 of the preceding section, elements can interact only through their outputs and inputs. Let E_1 and E_2 be two active elements, $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ their input vectors, and $\boldsymbol{y}^{(1)}$ and $\boldsymbol{y}^{(2)}$ their output vectors. Element E_1 can act on element E_2 only in such a way that E_2 "receive" as inputs the state of (all or some) outputs of E_1. For instance, element E_2 "receives" as inputs such output states of element E_1 as temperature and colour but not sounds. In other words, certain components of the output vector $\boldsymbol{y}^{(1)}$ become components of the input vector $\boldsymbol{x}^{(2)}$.

Denoting the components of the vector $\boldsymbol{y}^{(1)}$ by $y_1^{(1)}$, $y_2^{(1)}$, ..., $y_n^{(1)}$ and the components of the vector $\boldsymbol{x}^{(2)}$ by $x_1^{(2)}$, $x_2^{(2)}$, ..., $x_m^{(2)}$, we can write

$$x_j^{(2)} = y_i^{(1)} \tag{3.1}$$

for certain values of i and j, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Of course, equation (3.1) must be satisfied for at least one pair of values (i, j), for otherwise there would be no action of element E_1 on element E_2.

When components of the output vector of element E_1 are thus transformed into components of the input vector of element E_2, the two elements are said to be *coupled*.

The coupling of element E_1 with element E_2 is represented graphically as follows:

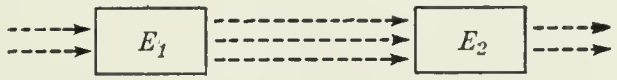

Fig. 3a

or:

Fig. 3b

In both figures all of the components of the input and output vectors are shown. In Fig. 3a element E_1 is coupled with element E_2 through all the components of its output vector, whereas in Fig. 3b, only through two components of that vector.

In practice it is rather inconvenient to use the individual components of the output and input vectors. With a large number of components the graphic picture of the coupling of elements becomes complicated. This picture, however, can be simplified in the following way. We introduce a square matrix which we denote by C_{12}, consisting of m or n rows and columns, depending on whether $m \geqslant n$ or $n \geqslant m$. The elements of the matrix have the value 1 for all i and j for which equation (3.1) is satisfied and they have the value 0 for all i and j for which the equation is not satisfied. In other words, an element of the matrix has the value 1 if the corresponding component of vector $y^{(1)}$ becomes components of vector $x^{(2)}$ and the value 0 if this is not the case. The matrix introduced has, therefore, the following shape (for example)

$$C_{12} = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 0 & \dots & 0 \end{bmatrix}. \tag{3.2}$$

This is a zero-one matrix with at least one element of the matrix equal to 1 and at most one element in each row (and column) equal to 1. If $m > n$, then $m - n$ rows consist of only zeros, and if $n > m$, then $n - m$ columns consist of only zeros; in that case $m - n$ inputs of element E_2 or $n - m$ outputs of element E_1 do not "participate" in the coupling. Matrix C_{12} shall be called the *coupling matrix* of element E_1 with element E_2.

With the aid of the coupling matrix, equation (3.1) can be rewritten in the form

$$x^{(2)} = C_{12}\, y^{(1)}, \tag{3.3}$$

i.e. in the form of a vector equation.

The graphic picture of the coupling of the element E_1 with the element E_2 can be simplified by drawing:

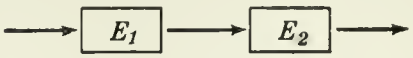

FIG. 4

This figure shows only vectors and not their individual components. The arrow denoting the coupling of the element E_1 with the element E_2 stands for the vector equation (3.3).

Now let us go on to consider the coupling of a larger number of active elements. We denote the individual elements by E_1, $E_2, E_3, \ldots$, and their input and output vectors, respectively by $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, ... and $y^{(1)}$, $y^{(2)}$, $y^{(3)}, \ldots$

If all the elements are coupled in pairs, then we say that there is a *chain of couplings*. Since the numeration of the elements is arbitrary, the chain of couplings can be represented as a sequence of vector equations:

$$\begin{aligned} x^{(2)} &= C_{12}\, y^{(1)}, \\ x^{(3)} &= C_{23}\, y^{(2)}, \\ &\cdots\cdots\cdots \end{aligned} \tag{3.4}$$

Here C_{12}, $C_{23}, \ldots$ denote the coupling matrices of element E_1 with element E_2, of element E_2 with element E_3, etc. A vector equation in the sequence of equalities (3.4) is called a *link* in the chain of couplings.

A chain of couplings is represented graphically as follows:

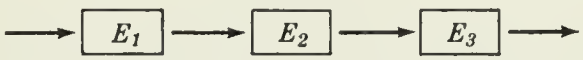

FIG. 5

We differentiate between *open* and *closed* chains of couplings called also *open* and *closed loops*. In a closed chain (loop) some active element is coupled with an element preceding it in the chain. Such coupling with an element appearing earlier in the chain of couplings is known as *feed-back*. A feature of a closed chain of couplings is that it has a *feed-back* among its links.

A chain which does not have any *feed-back* among its links is said to be *open* (an open loop).

Since the numeration of the elements is arbitrary, a closed chain of couplings can be expressed in the form of the vector equations

$$\begin{aligned} x^{(2)} &= C_{12}\, y^{(1)}, \\ x^{(3)} &= C_{23}\, y^{(2)}, \\ &\cdots\cdots\cdots \\ x^{(1)} &= C_{r1}\, y^{(r)}. \end{aligned} \tag{3.5}$$

The last vector equality in this case represents a *feed-back*. The feed-back occurs between element E_r and element E_1 (of course, we assume that $r > 1$).

A graphic picture of an open chain of couplings is given in Fig. 5. A closed chain is depicted (as an example) in Fig. 6. In this figure, there is a feed-back between element E_3 and element E_1.

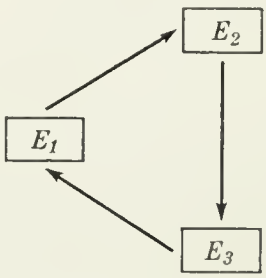

FIG. 6

An active element can be coupled with more than one other active element. Conversely, more than one element can be coupled with a given element. We say then that the couplings are *ramified*, there being *ramification of output* in the first case and *ramification of input* in the second.

In Fig. 7a we have an example of ramification of output and in Fig. 7b, an instance of ramification of input:

Ramification of output is expressed by the following vector equations

$$
\begin{aligned}
x^{(2)} &= C_{12}\, y^{(1)}, \\
x^{(3)} &= C_{13}\, y^{(1)}, \\
&\cdots\cdots\cdots \\
x^{(q+1)} &= C_{1,\,q+1}\, y^{(1)}.
\end{aligned}
\tag{3.6}
$$

Here, q denotes the number of elements coupled with the element E_1.

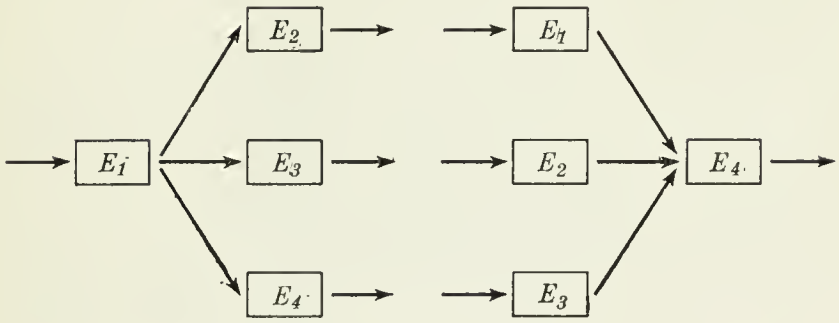

FIG. 7a FIG. 7b

In similar fashion ramification of input is expressed by the vector equations

$$
\begin{aligned}
x_1^{(q+1)} &= C_{1,\,q+1}\, y^{(1)}, \\
x_2^{(q+1)} &= C_{2,\,q+1}\, y^{(2)}, \\
&\cdots\cdots\cdots \\
x_q^{(q+1)} &= C_{q,\,q+1}\, y^{(q)}.
\end{aligned}
\tag{3.7}
$$

The left-hand side of these equations can be interpreted as forming the (vector) components of the compound vector $x^{(q+1)} = (x_1^{(+1)}, x_2^{(q+1)}, \ldots, x_q^{(q+1)})$.

In this case q stands for the number of elements with which the element E_{q+1} is coupled.

In a set containing a larger number of coupled active elements, there can be many different arrangements of couplings: open and closed chains (loops) and all sorts of ramifications. A set of such varied arrangements of couplings is called a *network of couplings*. A network of couplings can be expressed as a set of vector equalities of the type of equalities (3.4), (3.5), (3.6) and (3.7). It also can be represented graphically.

A graphic picture of a network of couplings is given in Fig. 8.

In this example, there is an open chain (loop) of couplings of elements E_1, E_2 and E_3, a closed chain (loop) of couplings of elements E_2, E_3 and E_4, and a ramification of output at element E_3 and a ramification of input at element E_7. Finally, element E_8 is not coupled with any element in the given set of acting elements. All this arrangement of couplings and absence of couplings constitutes a network of couplings as shown in the graphic picture.

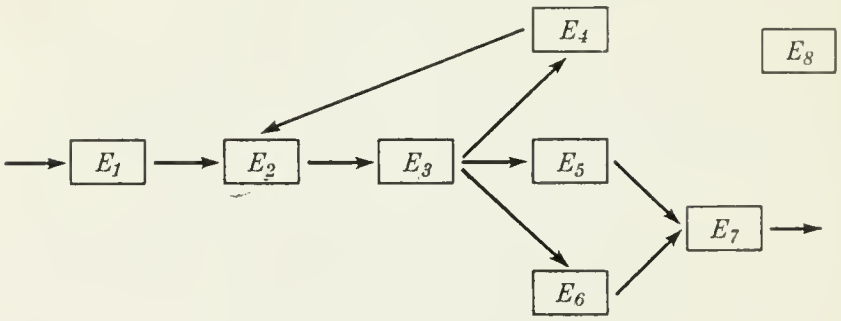

FIG. 8

Networks of couplings are a special case of so-called *graphs*, i.e. of abstract relations between the elements of a set. They are treated by a separate branch of mathematics known as the theory of graphs.[2] This theory is useful in investigation of the properties of networks of couplings of active elements.

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[2] The systematic mathematical study of graphs was initiated by Denes̆ Koenig, *Theorie der endlichen und unendlichen Graphen*, Leipzig, 1936. Hence, too, the term "graph". An exposition of this field as it stands today is given by Claude Berge, *Théorie des graphes et ses applications*, Paris, 1958, an English translation of which has now appeared (London, Methuen & Co., 1962).

IV. THE SYSTEM OF ACTIVE ELEMENTS AND ITS STRUCTURE

A SET of coupled active elements is called a *system of active elements* or, briefly, a *system*.[3] Each element of a system is coupled with at least one other element of the system or at least one other element in the system is coupled with the given element. In a system, therefore, there are no isolated elements, i.e. elements which are neither coupled with any other element nor have any other element coupled with them.

The network of couplings between elements of a system is called the *structure of the system.*

We assume that a system contains a finite number of elements. The number of elements in the system is denoted by N. The individual elements are denoted by E_1, E_2, ..., E_N. The input and output vectors are denoted by $x^{(1)}$, $x^{(2)}$, ..., $x^{(N)}$ and $y^{(1)}$, $y^{(2)}$, ..., $y^{(N)}$, respectively.

If an element E_r is coupled with element E_s, the vector equation[4]

$$x^{(s)} = C_{rs} y^{(r)} \tag{4.1}$$

is satisfied. In this equation, C_{rs} is the coupling matrix of element E_r with element E_s. As we know,[5] this is a square zero-one matrix in which at least one element is 1 and every row and every column contains at most one element equal to 1.

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[3] Generally speaking, "system" is taken to mean a set of elements together with the set of relations between the elements. The set of such relations (and of all their isomorphic transformations) is called the *structure* of the system. For instance, cf. Paul Lorenzen, *Einfuehrung in die operative Logik und Mathematik*, Berlin 1955, pp. 240–241. In our case, we have a set of active elements, the couplings of these elements are the relations.

[4] Cf. (3.3) above.

[5] Cf. (3.2) above.

We now generalize this matrix for all elements E_r and E_s regardless of whether E_r is coupled with E_s or not. This is done by writing, for the case when E_r is not coupled with E_s,

$$C_{rs} = 0. \tag{4.2}$$

In other words, in this case we define the coupling matrix as a zero matrix (i.e. one in which all the elements are zero). In this case when the active element E_r is not coupled with the active element E_s, therefore, the vector equality (4.1) takes on the form

$$0 = C_{rs}y^{(r)}. \tag{4.3}$$

Having generalized the coupling matrix C_{rs} in this way, we can write the vector equation (4.1) for every pair of elements E_r and E_s, regardless of whether E_r is coupled with E_s or not. We then have $N(N-1)$ vector equations

$$x^{(s)} = C_{rs}y^{(r)} \qquad (r, s = 1, 2, \ldots, N;\ r \neq s). \tag{4.4}$$

We assume that $r \neq s$ since matrix C_{rs} is always a zero matrix for $r = s$. By definition, the active element is coupled with other active elements. No element is coupled with itself.

The coupling matrices, thus generalized, together form a square matrix with N rows and N columns. Namely,

$$S = \begin{bmatrix} 0 & C_{12} & \ldots & C_{1N} \\ C_{21} & 0 & \ldots & C_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ C_{N1} & C_{N2} & \ldots & 0 \end{bmatrix}. \tag{4.5}$$

This matrix, which we denote by S, shall be called the *structure matrix of the system*. For it represents the structure of the system, i.e. the network of couplings of the elements of the system.

The matrix S is a zero-one matrix because all coupling matrices C_{rs}, which are its submatrices, are zero-one matrices.

If the coupling matrices are not all of the same order we border them with zeros wherever neeessary in order to make them "fit together" into the matrix S.

Since each element in the system is coupled with at least one other element or at least one other element is coupled with the given element, therefore, the minimum number of couplings in

the system is $N-1$. The matrix, therefore, contains at least $N-1$ non-zero submatrices C_{rs}. Moreover, these non-zero submatrices are so arranged that if the rth row of the matrix S contains only zero submatrices, the rth column must contain at least one non-zero submatrix. For if the element E_r is not coupled with any other element, then at least one other element must be coupled with the element E_r.

An element which is not coupled with any other element in the system or which has no other element coupled to it is called a *boundary element* of the system. An element in the system which is not a boundary element shall be called an *interior element* of the system. The set of boundary elements of a system shall be called the *surface* of the system and the set of the interior elements of the system shall be known as the *interior* of the system. For the sake of illustration, let us look at Fig. 8. The active elements E_1, E_2, E_3, E_4, E_5, E_6, E_7 form a system; the isolated element E_8 does not belong to it. Elements E_1 and E_7 are boundary elements of the system; the set of them constitutes the surface of the system. On the other hand, elements E_2, E_3, E_4, E_5, E_6 are interior elements of the system; the set of them constitutes the interior of the system.

Among the boundary elements of the system we distinguish between those which are not coupled with other elements in the system and those with which no other elements in the system are coupled. In the case of the first-mentioned, the components of the output vector are not components of the input vector of any other element in the system. We shall call them the *output boundary elements*. In the case of the last-mentioned, the components of the input vector are not components of the output vector of any other element in the system. We shall call them the *input boundary elements*. In Fig. 8, E_1 is a input boundary element whereas E_7 is an output boundary element.

The input boundary and output boundary elements are expressed in the structure matrix of the system. If element E_r is an input boundary element, then the rth column of the matrix S contains only zero submatrices. For, in keeping with the generalized definition of the coupling matrix, we then have

$$C_{1r} = C_{2r} = \ldots = C_{Nr} = 0.$$

If, on the other hand, element E_r is an output boundary element, then in accordance with the definition we have

$$C_{r1} = C_{r2} = \ldots = C_{rN} = 0.$$

In this case, the rth row of the matrix S consists of only zero submatrices. The converse is also true; a column in the matrix S consisting of only zero submatrices signifies that the corresponding element is an input boundary element and a row of the matrix S consisting of only zero submatrices signifies that the corresponding element is an output boundary element. It follows that the set of all the columns and rows of the matrix S consisting of only zero submatrices corresponds to the system's surface. On the other hand, the columns and rows containing at least one non-zero submatrix C_{rs} correspond to the interior of the system. A system which does not contain any boundary elements, i.e. does not possess a surface, shall be called a *closed system*. From what has been said it follows that the structure matrix of a closed system contains neither columns nor rows consisting of only zero submatrices; each column and each row contains at least one non-zero submatrix. In a closed system, therefore, there are at least N couplings. A system which contains boundary elements shall be called an *open system*. The structure matrix of an open system contains columns or rows (or both) consisting of only zero submatrices and representing together the surface of the system.

The structure matrix of the system reflects also other properties of the couplings of the elements of the system. Feedbacks, for instance, are all represented by submatrices C_{rs} which lie below the diagonal of the matrix. Feed-backs are couplings with an element preceding the given element in the chain of couplings. The corresponding couplings matrices C_{rs} have $s < r$. This determines their position below the diagonal.

A closed system possesses at least one feed-back. For let us assume that there are no feed-backs in the system. Then the matrix of the system is

$$S = \begin{bmatrix} 0 & C_{12} & \dots & C_{1N} \\ 0 & 0 & \dots & C_{2N} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}. \tag{4.6}$$

As can be seen, there is then at least one input boundary element and at least one output boundary element. No element is coupled with element E_1 and element E_n is not coupled with any element (we recall that the numeration of the elements is arbitrary). The system, therefore, cannot be a closed one.

An open chain of couplings (open loop) is expressed by a structure matrix of the system in which all submatrices C_{rs} below the diagonal are zero and, moreover, in each row (apart from the last one which consists entirely of zero submatrices) as well as in each column (apart from the first one which consists entirely of zero submatrices) there is exactly one non-zero submatrix C_{rs}. In other words, the structure matrix of the system has the following appearance

$$S = \begin{bmatrix} 0 & C_{12} & 0 & \dots & 0 & 0 \\ 0 & 0 & C_{23} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & C_{N-1,N} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{4.7}$$

If the chain of couplings is closed (closed loop), then in (4.7) there is at least one non-zero submatrix below the diagonal.

If a row of the structure matrix of the system contains more than one non-zero submatrix C_{rs}, this indicates a ramification of output. For instance, let there be non-zero submatrices C_{12}, C_{13}, C_{14} in the first row of the matrix S; then element E_1 is coupled with elements E_2, E_3 and E_4. Similarly, if a column of the structure matrix contains more than one non-zero submatrix C_{rs}, this is an indication of a ramification of input. For instance, in matrix (4.6) if C_{1N} and C_{2N} are non-zero, element E_1 and element E_2 are coupled with element E_N.

Thus properties of the network of couplings of the system can be read from the structure matrix.

V. SYSTEMS OF HIGHER ORDER

CONSIDER two systems of active elements and denote them by U_1 and U_2. Let the system U_1 contain N_1 elements denoted by $E_1, E_2, \ldots, E_{N_1}$, and the system U_2 contain N_2 elements denoted by $E_{N_1+1}, E_{N_1+2}, \ldots, E_{N_1+N_2}$.

If no element of the system U_1 is coupled with any element of the system U_2, nor any element of the system U_2 with any element of the system U_1, we say that these two systems are *independent*. If, on the other hand, at least one element of the system U_1 is coupled with at least one element of the system U_2, or at least one element of the system U_2 is coupled with at least one element of the system U_1, both systems form a *new system* which we denote by U'. Such a system, formed through the coupling of elements of two or more systems, shall be called a *system of the second order.*

The structure matrix of the system U'—let us denote it by S'—is:

$$S' = \left[\begin{array}{ccc:ccc} 0 & C_{12} & \ldots C_{1N_1} & C_{1,N_1+1} & C_{1,N_1+2} & \ldots C_{1,N_1+N_2} \\ C_{21} & 0 & \ldots C_{2N_1} & C_{2,N_1+1} & C_{2,N_1+2} & \ldots C_{2,N_1+N_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ C_{N_1 1} & C_{N_1 2} & \ldots 0 & C_{N_1,N_1+1} & C_{N_1,N_1+2} & \ldots C_{N_1,N_1+N_2} \\ \hdashline C_{N_1+1,1} & C_{N_1+1,2} & \ldots C_{N_1+1,N_1} & 0 & C_{N_1+1,N_1+2} & \ldots C_{N_1+1,N_1+N_2} \\ C_{N_1+2,2} & C_{N_1+2,2} & \ldots C_{N_1+2,N_1} & C_{N+2,N_1+1} & 0 & \ldots C_{N_1+2,N_1+N_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ C_{N_1+N_2,1} & C_{N_1+N_2,2} & \ldots C_{N_1+N_2,N_1} & C_{N_1+N_2,N_1+1} & C_{N_1+N_2,N_1+2} & \ldots 0 \end{array}\right] \tag{5.1}$$

As can be seen (this is shown by formula (5.1)), the matrix S' can be regarded as consisting of four submatrices, which, in turn, consist of submatrices C_{rs} representing the couplings of the

elements in the system U'. Those four submatrices shall be denoted by S_{11}, S_{12}, S_{21}, and S_{22} so that the matrix S' ean be written in the form

$$S' = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}. \tag{5.2}$$

The submatrix S_{11} is the strueture matrix of the system U_1 and the submatrix S_{22} is the structure matrix of the system U_2. The submatrix S_{12} rcpresents the couplings of elements of the systcm U_1 with elemcnts of the systcm U_2. The submatrix S_{21}, on thc other hand, expresses the couplings of elements of the systcm U_2 with elements of the system U_1, in other words, the fced-backs of the elements of the two systems.

From the matrix S' it is possible to read off the properties of the second-order system U'. First of all, S_{12} and S_{21} cannot both be zero matriccs. For in that event we would have

$$S' = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}, \tag{5.3}$$

i.e. no elcment of the systcm U_1 would be coupled with any elcment of the system U_2, nor would any element of the system U_2 be coupled with any element of the system U_1. The systcms U_1 and U_2 would thus be independent. If S_{12} is not a zero matrix, and S_{21} is, elements of the system U_1 are coupled with elements of the system U_2 but no clements of the system U_2 are coupled with elements of the system U_1. The revcrsc is true when $S_{21} \neq 0$ and $S_{12} = 0$. We may express this by saying that in the first ease the system U_1 aets unidirectionally on the system U_2 while in the seeond ease the system U_2 acts unidireetionally on the system U_1.

Finally, if U' is to be really a system of the second order consisting of two systems U_1 and U_2 then S_{22} cannot be equal to zero. For in that ease only thc system U_1 would exist. If, at the same time, S_{12} and S_{21} werc not zcro, this would mean that the system U_1 is *expanded* by the incorporation of the additional elcmcnts E_{N_1+1}, E_{N_1+2}, ..., $E_{N_1+N_2}$. But these elements are not

coupled with each other, in other words, they do not constitute a system. The coupling of elements E_{N_1+1}, E_{N_1+2}, ..., $E_{N_1+N_2}$ is expressed in the fact that $S_{22} \neq 0$.

A system of the second order can also come into being through the coupling of a larger number of systems. For instance, if we have three systems U_1, U_2, U_3 it is possible, by coupling their elements, to create a new system U' which has a structure matrix of the form

$$S' = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}. \tag{5.4}$$

Here, the submatrices S_{11}, S_{22}, S_{33} are the structure matrices of the systems U_1, U_2, U_3, respectively. The remaining submatrices represent the coupling of those systems. Obviously, not all of the submatrices which are not on the diagonal can be zero, for in that case the systems U_1, U_2, U_3 would be independent and U' would not be a system.

By coupling the elements of two or more systems of the second order, U_1', U_2', U_3', etc., we obtain a *system of the third order*. We denote its structure matrix by S''. If, for instance, a third-order system U'' consists of two systems of the second order, its structure matrix can be represented in the form

$$S'' = \begin{bmatrix} S_{11}' & S_{12}' \\ S_{21}' & S_{22}' \end{bmatrix}. \tag{5.5}$$

In this formula, the submatrices S_{11}' and S_{22}' are the structure matrices of the systems U_1' and U_2', the submatrix S_{12}' represents the coupling, of elements of the system U_1' with elements of the system U_2', and the submatrix S_{21}' expresses the couplings of elements of the system U_2' with elements of the system U_1'.

Replacing the submatrices in (5.5) by expressions of the type (5.2), we obtain the matrix S'' as a matrix consisting of 16 submatrices of the type S_{ij} $(i, j = 1, 2, 3, 4)$. In this way the third-order system U'' consisting of two second-order systems

can be represented as a second-order system consisting of four first-order systems.

Generally, if k_r denotes the number of $(r-1)$-order systems forming a system of the rth order, then, as can be easily seen, a system of the qth order can be represented as a system of the second order, consisting of k_1, k_2, ..., k_q first-order systems.

VI. THE MODE OF ACTION OF A SYSTEM

In a system containing N active elements, the inputs and outputs of these elements satisfy $N(N-1)$ vector equations (4.4), i.e.

$$x_r^{(s)} = C_{rs}\, y^{(r)} \qquad (r, s = 1, 2, \ldots, N;\ r \neq s). \tag{6.1}$$

These equations represent the couplings of the elements. The subscript r on the left-hand side indicates that $x_r^{(s)}$ is the state of inputs of E_s due to being coupled with E_r.

Each element of the system possesses a definite mode of action expressed mathematically by transformation (2.4). Denoting by T_r the transformation expressing the mode of action of the element E_r, we have N such transformations:

$$y_r^{(r)} = T_r\,(x^{(r)}) \qquad (r = 1, 2, \ldots, N). \tag{6.2}$$

Substituting the transformations (6.2) into the vector equations (6.1), we obtain

$$x_r^{(s)} = C_{rs}\, T_r\,(x^{(r)}) \qquad (r, s = 1, 2, \ldots, N;\ r \neq s). \tag{6.3}$$

This is a set of $N(N-1)$ transformations. Writing

$$R_{rs} = C_{rs}\, T_r\,, \tag{6.4}$$

we can present these transformations symbolically as

$$x_r^{(s)} = R_{rs}\,(x^{(r)}) \qquad (r, s = 1, 2, \ldots, N;\ r \neq s). \tag{6.5}$$

Upon subjecting the vector equations (6.1) to the transformations (6.2) we obtain

$$T_s\,(x_r^{(s)}) = T_s\, C_{rs}\,(y^{(r)}),$$

and consequently

$$y_r^{(s)} = T_s\, C_{rs}\,(y^{(r)}) \quad (r, s = 1, 2, \ldots, N;\ r \neq s). \tag{6.6}$$

Thus we obtain again a set of $N(N-1)$ transformations. Writing

$$P_{rs} = T_s\, C_{rs}\,, \tag{6.7}$$

we can express these transformations symbolically in the form

$$y_r^{(s)} = P_{rs}(y^{(r)}) \quad (r, s = 1, 2, \ldots, N;\ r \neq s). \tag{6.8}$$

The set of transformations (6.5) assigns new values of the input vectors of the system's elements to given initial values of these vectors. Similarly, the set of transformations (6.8) assigns new values of the output vectors of the system's elements to given initial values of these vectors. The sets of transformations (6.5) and (6.8) express, as we say, the *mode of action of the system.* They indicate in what manner a set of given states of the inputs and outputs of the elements of a system is transformed into a new set of states of those inputs and outputs. To employ an expression of Karl Marx, it can be said that the two sets of transformations express "the inner law of motion of the system".[6]

We now introduce the *vector of the input states of the system* as a compound vector, its components being the input vectors of the elements belonging to the system. Denoting this vector by $\boldsymbol{X}$, we have

$$\boldsymbol{X} = (x^{(1)}, x^{(2)}, \ldots, x^{(N)}). \tag{6.9}$$

We introduce in a similar way a *compound vector of the output states of the system.* We denote this vector by $\boldsymbol{Y}$ and write

$$\boldsymbol{Y} = (y^{(1)}, y^{(2)}, \ldots, y^{(N)}). \tag{6.10}$$

These compound vectors are equivalent to the quasi-diagonal matrices[7]

$$X = \begin{bmatrix} x^{(1)} & 0 & \ldots & 0 \\ 0 & x^{(2)} & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & x^{(N)} \end{bmatrix} \tag{6.9a}$$

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[6] Cf. Karl Marx, *Capital*, Vol. I, Foreign Languages Publishing House, Moscow, 1954, p. 10.

[7] A quasi-diagonal matrix is one in which submatrices or vectors appear along the diagonal and all the remaining elements are equal to zero. For a definition of a quasi-diagonal matrix, see V. I. Smirnov, *Kurs vysshei matematiki* (in Russian), Vol. III Part 1, 4th ed., Leningrad–Moscow, 1949, p. 104; English translation published by Pergamon Press, 1964.

and

$$Y = \begin{bmatrix} y^{(1)} & 0 & \dots & 0 \\ 0 & y^{(2)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & y^{(N)} \end{bmatrix}. \tag{6.10a}$$

Denoting by m_r the number of inputs and by n_r the number of outputs of the element E_r $(r = 1, 2, \dots, N)$, we find that the matrix X has N rows and $\sum_{r=1}^{N} m_r$ columns and the matrix Y has N rows and $\sum_{r=1}^{N} n_r$ columns. As need be, X and Y can be interpreted as either compound vectors (6.9) and (6.10) or as quasi-diagonal matrices (6.9a) and (6.10a).

The set of transformations (6.5) is equivalent to the following matrix equation

$$\begin{bmatrix} 0 & x_1^{(2)} & \dots & x_1^{(N)} \\ x_2^{(1)} & 0 & \dots & x_2^{(N)} \\ \dots & \dots & \dots & \dots \\ x_N^{(1)} & x_N^{(2)} & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & R_{12}(x^{(1)}) & \dots & R_{1N}(x^{(1)}) \\ R_{21}(x^{(2)}) & 0 & \dots & R_{2N}(x^{(2)}) \\ \dots & \dots & \dots & \dots \\ R_{N1}(x^{(N)}) & R_{N2}(x^{(N)}) & \dots & 0 \end{bmatrix}. \tag{6.11}$$

Similarly, the set of transformations (6.8) is equivalent to the matrix equation

$$\begin{bmatrix} 0 & y_1^{(2)} & \dots & y_1^{(N)} \\ y_2^{(1)} & 0 & \dots & y_2^{(N)} \\ \dots & \dots & \dots & \dots \\ y_N^{(1)} & y_N^{(2)} & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & P_{12}(y^{(1)}) & \dots & P_{1N}(y^{(1)}) \\ P_{21}(y^{(2)}) & 0 & \dots & P_{2N}(y^{(2)}) \\ \dots & \dots & \dots & \dots \\ P_{N1}(y^{(N)}) & P_{N2}(y^{(N)}) & \dots & 0 \end{bmatrix}. \tag{6.12}$$

The columns on the left-hand side of (6.11) are compound vectors which indicate the new state of inputs due to the coupling

of the elements. In the same way the columns on the left-hand side of (6.12) indicate the new state of the outputs due to the coupling of the elements. Denoting these columns by $\overline{x}^{(1)}$, $\overline{x}^{(2)}$, ..., $\overline{x}^{(N)}$ and $\overline{y}^{(1)}, \overline{y}^{(2)}, \ldots, \overline{y}^{(N)}$ respeetively, the left-hand side's of (6.11) and (6.12) can be written as compound veetors

$$\overline{X'} = (\overline{x}^{(1)}, \overline{x}^{(2)}, \ldots, \overline{x}^{(N)})$$

and

$$\overline{Y'} = (\overline{y}^{(1)}, \overline{y}^{(2)}, \ldots, \overline{y}^{(N)}).$$

We thus write the matrix equality (6.11), or the equivalent set of transformations (6.5), in the form

$$X' = R(X), \tag{6.13}$$

and the matrix equality (6.12), or the equivalent set of transformations (6.8), we write in the form

$$Y' = P(Y). \tag{6.14}$$

Here, R and P are transformation operators, X and Y represent the initial state of the inputs and outputs of all the elements of the system, whereas X' and Y' denote the new state of the inputs and outputs after the transformation.

It suffiees in to consider either transformation (6.13) or transformation (6.14). For from the veetor equations (6.1) we obtain the matrix equation

$$\begin{bmatrix} 0 & x_1^{(2)} & \ldots & x_1^{(N)} \\ x_2^{(1)} & 0 & \ldots & x_2^{(N)} \\ \ldots & \ldots & \ldots & \ldots \\ x_N^{(1)} & x_N^{(2)} & \ldots & 0 \end{bmatrix} = \begin{bmatrix} 0 & C_{12}y^{(1)} & \ldots & C_{1N}y^{(1)} \\ C_{21}y^{(2)} & 0 & \ldots & C_{2N}y^{(2)} \\ \ldots & \ldots & \ldots & \ldots \\ C_{N1}y^{(N)} & C_{N2}y^{(N)} & \ldots & 0 \end{bmatrix}. \tag{6.15}$$

The right-hand side of this equality ean be written in the form YS where Y is the quasi-diagonal matrix (6.10a), equivalent to the eompound vector (6.10), and S is the structure matrix of the system

$$S = \begin{bmatrix} 0 & C_{12} & \ldots & C_{1N} \\ C_{21} & 0 & \ldots & C_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ C_{N1} & C_{N2} & \ldots & 0 \end{bmatrix}. \tag{6.16}$$

The left-hand side of the above matrix equality is equivalent to the eompound vector X'. As a result, we have

$$X' = YS. \tag{6.17}$$

The transformations (6.13) and (6.14), therefore, follow one from the other. The operators R and P can be presented symbolically in the form of a matrix of operators appearing in the sets of transformations (6.5) and (6.8). Namely,

$$R = \begin{bmatrix} 0 & R_{12} & \dots & R_{1N} \\ R_{21} & 0 & \dots & R_{2N} \\ \dots & \dots & \dots & \dots \\ R_{N1} & R_{N2} & \dots & 0 \end{bmatrix} \tag{6.18}$$

and

$$P = \begin{bmatrix} 0 & P_{12} & \dots & P_{1N} \\ P_{21} & 0 & \dots & P_{2N} \\ \dots & \dots & \dots & \dots \\ P_{N1} & P_{N2} & \dots & 0 \end{bmatrix}. \tag{6.19}$$

Let us denote by T the diagonal matrix

$$T = \begin{bmatrix} T_1 & 0 & \dots & 0 \\ 0 & T_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & T_N \end{bmatrix}. \tag{6.20}$$

The non-zero elements of this matrix constitute the transformation operators corresponding to the individual elements of the system. This diagonal matrix shall be called the *matrix of modes of action of the elements.*

If (6.4) and (6.7) are taken into account the operators R and P can be presented as products, viz.

$$R = TS \tag{6.21}$$

and

$$P = ST, \tag{6.22}$$

where S is the structure matrix of the system (6.16). (Cf. also (4.7) above.)

In consequence it shows that the mode of action of the system, expressed mathematically by the transformation (6.11) or (6.12), can be presented in the form

$$\boldsymbol{X}' = TS(\boldsymbol{X}), \tag{6.23}$$

$$\boldsymbol{Y}' = ST(\boldsymbol{Y}). \tag{6.24}$$

As can be seen, the mode of action of the system depends both on T, the matrix of modes of action of the elements, as well as on the structure matrix S of the system. In order to de-

termine the mode of action of the system it is not enough to know the modes of action of the elements, which are expressed mathematically by the matrix T, but it is necessary also to know the structure matrix S of the system which represents the network of couplings of the elements.

A system of active elements, therefore, is endowed with its own specific mode of action, "a law of motion", consisting in the transformation of the input and output states of the elements. This mode of action is a feature of the system *as a whole* for it depends not only on the mode of action of the individual elements but also on the structure of the system, i.e. on the network of couplings of the elements. This is expressed mathematically in (6.23) and (6.24). It is the structure represented by the matrix S that endows the system with the quality of a whole. The same elements with the same modes of action but coupled in a different manner constitute a different system with a different mode of action of the system. In mathematical language: with an unaltered matrix T of modes of action of the elements, the mode of action becomes different if there is a different structure matrix S of the system. A difference in the structure implies a difference in the system's mode of action.

Systems of a higher order can also be shown to have new properties, their own mode of action, which depends not only on the mode of action of the first-order systems forming the system of higher order. The mode of action of a higher-order system depends also on its structure, i.e. on the manner in which elements of different first-order systems are coupled with each other. This manner is expressed in matrices S_{12} and S_{21} and in similar matrices appearing in (5.2) and (5.4).

In this way an explanation is provided for the question of wholes with modes of action which cannot be derived from the mere modes of action of the individual elements. Such wholes are simply sets of coupled active elements, in other words, i.e. systems. Their mode of action constitutes the combined result of the modes of action of the individual elements, and of the structure of the system, i.e. of the network of couplings of the elements of which the system is composed.

VII. THE PROCESS OF DEVELOPMENT OF A SYSTEM

We now introduce explicitly the assumption that the action of the system's elements proceeds in time. A certain time elapses between a change in the state of the inputs of an element and a change in its outputs. We call this the *reaction time*.[8]

The action may be sudden or gradual. In the case of a sudden process, a certain time after the change in the input state there is a single change in the output state, e.g. ten minutes after the cooker burner is lit the tea-kettle begins to boil. In the case of a gradual process, changes in the output state occur in a certain interval of time, after the change in the input state has taken place proceeding discretely or continuously. For instance, once the burner is lit the temperature of the water in the tea-kettle gradually rises until it reaches the boiling level ten minutes later (continuous process), or after a nerve has been irritated the muscle involved contracts at one-second intervals during a period of two minutes (discrete process). In the case when the action is sudden, the reaction time is denoted by a single (positive) real number, e.g. ten minutes; in the case of a gradual discrete process, a sequence of real numbers (e.g. 1, 2, ..., 10 sec.) is required to denote the reaction time and in the case of a gradual continuous process a continuous interval of real numbers is required to denote the reaction time, e.g. the interval from 0 to 10 sec.

This time process must now be taken into account in the transformations (6.2) which are the mathematical expression of the mode of action of the elements of the system. For this purpose, we date the states of the inputs and outputs of each element. We denote by

$$x_t^{(r)} = (x_{1t}^{(r)}, x_{2t}^{(r)}, \ldots, x_{mrt}^{(r)}) \tag{7.1}$$

[8] Cf. Greniewski, *Cybernetics without Mathematics*, pp. 15–17.

the vector expressing the input states of the element E_r at the moment t. If the action is sudden, the vector determines the states of the outputs of the element after the reaction times $\theta_1^{(r)}$, $\theta_2^{(r)}$, ..., $\theta_{n_r}^{(r)}$, corresponding to the various outputs.[9] The output states are then expressed by the vector

$$\boldsymbol{y}_{t+\theta^{(r)}}^{(r)} = (y_{1,t+\theta_1^{(r)}}^{(r)},\ y_{2,t+\theta_2^{(r)}}^{(r)},\ \ldots,\ y_{n_r,t+\theta_{n_r}^{(r)}}^{(r)}). \tag{7.2}$$

As can be seen, the individual components of this vector refer in general to different moments of time $t + \theta_1^{(r)}$, $t + \theta_2^{(r)}$, ..., $t + \theta_{n_r}^{(r)}$. In special cases some, or all, of the moments of time may be the same; e.g. $\theta_1^{(r)} = \theta_2^{(r)}$, etc.

As a result, the transformations (6.2) now acquire the form

$$\boldsymbol{y}_{t+\theta^{(r)}}^{(r)} = T_r(\boldsymbol{x}_t^{(r)}) \quad (r = 1,\ 2,\ \ldots,\ N). \tag{7.3}$$

In view of this, the transformations (6.5) expressing the system's "law of motion" (i.e. mode of action) have the form

$$\boldsymbol{x}_{r,t+\theta^{(r)}}^{(s)} = R_{rs}(\boldsymbol{x}_t^{(r)}) \quad (r,\ s = 1,\ 2,\ \ldots, N;\ r \neq s). \tag{7.4}$$

The set of these transformations can be written as a transformation of a compound vector of the input states of all the elements of the system:

$$\boldsymbol{X}_{t+\theta} = R\,(\boldsymbol{X}_t) \tag{7.5}$$

or

$$\boldsymbol{X}_{t+\theta} = TS\,(\boldsymbol{X}_t). \tag{7.6}$$

This corresponds to formulae (6.13) and (6.23) of the preceding chapter. The compound vectors $\boldsymbol{X}_t$ and $\boldsymbol{X}_{t+\theta}$ are here defined as consisting of the components of $x_{it}^{(r)}$ and $x_{i,t+\theta_i^{(r)}}^{(r)}$ where $r = 1$, 2, ..., N and $i = 1$, 2, ..., m_r, i.e. of the components of the input vectors of all the elements in the system.

In a similar way, we find that the transformations (6.8) now acquire the form

$$\boldsymbol{y}_{r,t+\theta^{(s)}}^{(s)} = P_{rs}(\boldsymbol{y}_t^{(r)}) \quad (r,\ s = 1,\ 2,\ \ldots,\ N;\ r \neq s). \tag{7.7}$$

[9] The reaction times $\theta_1^{(r)}$, $\theta_2^{(r)}$, ..., $\theta_{n_r}^{(r)}$ of the individual components of the vector $\boldsymbol{y}_{t+\theta^{(r)}}^{(r)}$ are defined with respect to the whole vector $\boldsymbol{x}_t^{(r)}$ and not with respect to its individual components. The superscript $^{(r)}$ is required in order to show to which element in the system the reaction times refer.

Writing them as a transformation of a compound vector of the output states of all the elements of the system we have

$$\boldsymbol{Y}_{t+\theta} = P(\boldsymbol{Y}_t) \tag{7.8}$$

or

$$\boldsymbol{Y}_{t+\theta} = ST(\boldsymbol{Y}_t) \tag{7.9}$$

as corresponding to formulae (6.13) and (6.24) of the preceding chapter. The compound vectors $\boldsymbol{Y}_t$ and $\boldsymbol{Y}_{t+\theta}$ consist of the components of $y_{it}^{(r)}$ and $y_{i,\,t+\theta_i}^{(s)}$.

Thus the "law of motion" of the system now becomes expressed mathematically as a vectorial difference equation. This equation establishes the relations between the state of inputs (or outputs) of the system's elements at a certain moment t and the state of inputs (or outputs) at later moments $t + \theta_i^{(r)}$ (or $t + \theta_i^{(s)}$) corresponding to the various reaction times. The "law of motion" of the system, therefore, becomes a law which determines the changes in the states of inputs (or outputs) taking place in the course of time. The mode of action of the system acquires the character of a process of the *system's development in time.*

The process of the system's development in time, or, as we shall say henceforth, the process of the system's development is obtained by the solution of the above difference equation. The solution can be obtained by iteration. Denoting the initial moment by $t = 0$, and substituting successively the left-hand side of (7.5) for the vector on the right-hand side, we obtain

$$\boldsymbol{X}_\theta = R(\boldsymbol{X}_0), \quad \boldsymbol{X}_{2\theta} = R^2(\boldsymbol{X}_0), \quad \ldots,$$

etc.

Thus, in general,

$$\boldsymbol{X}_{k\theta} = R^k(\boldsymbol{X}_0) \quad (k \text{ is an integer}) \tag{7.10}$$

where R^k stands for k repetitions of the operation R.

When this substitution is made it is necessary to assume each time that the action becomes completely exhausted with time, that is, that the full time of action is equal to the longest reaction time $\theta_i^{(r)}$. For it is only after this time has elapsed that the state of all outputs is fully determined, and in turn deter-

mines—by means of the couplings of the elements—the new value of the compound vector of inputs of the system's elements. We assume, therefore, that

$$\theta = \max \theta_i^{(r)}. \tag{7.11}$$

In a similar way we find, by making successive substitutions in (7.8), that

$$\boldsymbol{Y}_{k\theta} = P^k(\boldsymbol{Y}_0), \tag{7.12}$$

where $\theta = \max \theta_i^{(s)}$.

Therefore, if all the input or all the output states of the system's elements are given at the initial moment $t = 0$, it is possible to determine the state of those inputs and outputs at the moments $t = k\theta$, corresponding to multiples of the longest reaction time. In other words, the state of the system at the initial moment determines its state at all later moments which are a multiple of the longest reaction time.

The difference equation (7.5) or (7.8) shall be called the *system's law of motion in time*. The solution of this equation, (7.10) or (7.12), shall be called the *law of development of the system*.

In deriving the system's law of motion in time and the law of development of the system, we assumed that the action is sudden. In the case of a gradual action in time, the mathematical form of those laws is similar but somewhat more complicated.

If the action in time is gradual a change in the state of the input vector at the moment t causes gradual changes in the output vector at later moments. At every moment, $t + \tau$, the output vector has a definite value which depends not only on the value of the input vector at the moment t but also on the length of time τ. For instance, the temperature of water in the tea-kettle depends not only on the temperature of the burner lit at the moment t, but also on how long that burner has been on. Thus we can write

$$y_{t+\tau}^{(r)} = T_r(x_t^{(r)}, \tau).$$

Here, τ is not greater than the longest reaction time $\theta_i^{(r)}$ since after that time the output vector ceases to change. We denote this by writing

$$0 \leqslant \tau \leqslant \max \theta_i^{(r)}.$$

Changing the starting point for counting time, i.e. writing t in place of $t + \tau$ and $t - \tau$ in place of t, we obtain

$$y_t^{(r)} = T_r(x_{t-\tau}^{(r)}, \tau). \tag{7.13}$$

The value of the output vector at the moment t depends on the value of the input vector at the moment $t - \tau$ and on the length of time τ.

In view of the gradual character of the action, however, the value of the output vector at the moment t depends on the value of the input vector not only at the single moment $t - \tau$, but its values at all the moments in the time interval $[t, t - \theta]$. The gradual effects of the action of the input states are superimposed, their sum determines the state of the outputs at the moment t. As a result we have

$$y_t^{(r)} = \sum_{\tau=0}^{\theta} T_r(x_{t-\tau}^{(r)}, \tau), \tag{7.14a}$$

where for the sake of abbreviation we write $\theta = \max \theta_i^{(r)}$.

If the gradual action is a discrete process, then τ assumes a finite number of values corresponding to the successive "time jumps" of the action. On the other hand, if this process is continuous, then τ assumes all values in the continuous interval of time $[0, \theta]$ and the right-hand side of equation (7.14) is an integral. We then have

$$y_t^{(r)} = \int_0^{\theta} T_r(x_{t-\tau}^{(r)}, \tau)\, d\tau. \tag{7.14b}$$

Inserting expressions (7.14a) and (7.14b) into (6.1), we obtain the transformations

$$x_{r,t}^{(s)} = C_{rs} \sum_{\tau=0}^{\theta} T_r(x_{t-\tau}^{(r)}, \tau) \tag{7.15a}$$

or

$$x_{r,t}^{(s)} = C_{rs} \int_0^{\theta} T_r(x_{t-\tau}^{(r)}, \tau)\, d\tau. \tag{7.15b}$$

From this, by making use of the notation (6.4), we obtain

$$x_{r,t}^{(s)} = \sum_{\tau=0}^{\theta} R_{rs}(x_{t-\tau}^{(r)}, \tau) \tag{7.16a}$$

or

$$x_{r,t}^{(s)} = \int_0^{\theta} R_{rs}(x_{t-\tau}^{(r)}, \tau)\ \mathrm{d}\tau, \tag{7.16b}$$

where $r, s = 1, 2, \ldots, N$ and $r \neq s$.

The set of these transformations can be presented as the transformation of the compound vector of the input states of the system, i.e.

$$X_t = \sum_{\tau=0}^{\theta} R(X_{t-\tau}, \tau) \tag{7.17a}$$

or

$$X_t = \int_0^{\theta} R(X_{t-\tau}, \tau)\ \mathrm{d}\tau. \tag{7.17b}$$

The vector X_t consists of the components $x_{it}^{(s)}$ and the vector $X_{t-\tau}$ consists of components $x_{i,t-\tau}^{(r)}$, where $r, s=1, 2, \ldots, N$ and $i = 1, 2, \ldots, m_r$. The symbol θ denotes the largest value of $\theta_i^{(r)}$.

The first of these vector transformations refers to the case of discrete gradual action. It constitutes a vectorial difference equation of higher order. The second vector transformation refers to the case of continuous gradual action and has the form of a vectorial integral equation.

In a similar way we obtain a vectorial difference equation or a vectorial integral equation

$$Y_t = \sum_{\tau=0}^{\theta} P(Y_{t-\tau}, \tau) \tag{7.18a}$$

and

$$Y_t = \int_0^{\theta} P(Y_{t-\tau}, \tau)\ \mathrm{d}\tau. \tag{7.18b}$$

In view of (6.17) these equations are equivalent to the equations (7.17a) and (7.17b).

The above equations express the system's law of motion in time. The solution of these equations yields vector functions of time X_t and Y_t which determine the development in time of the input and output states of the system. These functions

express the law of development of the system. In the case of discrete action, the shape of these functions i.e. the development of the system in time, depends on their values, i.e. on the input and output states, at a certain finite number of initial moments contained in the initial interval of time $[0, \theta]$. In the case of continuous action, however, it depends on all the values of these functions in the continuous initial time interval $[0, \theta]$. Thus, if the states of the system are given for the initial moments or for the initial interval of time, the entire further development of the system is determined.

VIII. EQUILIBRIUM AND STABILITY SELF-REGULATION

Particular attention is due to the case when $X_t = \text{const.}$ and $Y_t = \text{const.}$, i.e. when the state of the system does not change in time. We then say that the system is in a *state of equilibrium.*[10] In a state of equilibrium the input and output states of the system's elements are invariant in time. When the system is in equilibrium, equation (7.17) expressing the law of motion of the system assumes the form[11]

$$X = \sum_{\tau=0}^{\theta} R(X, \tau) \tag{8.1a}$$

or

$$X = \int_{0}^{\theta} R(X, \tau)\, d\tau. \tag{8.1b}$$

If the action is sudden then, as can be seen from (7.5), this form is even simpler, viz.

$$X = R(X). \tag{8.1c}$$

A similar form is assumed by the equivalent equation (7.18) or (7.8). In this equation the subscripts of Y simply have to be eliminated.

As can be seen, when the system is in equilibrium its law of motion assumes the form of an ordinary vector equation. This equation shall be called the *equilibrium equation.* The vector which is a solution of this equation, if it exists, determines the

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[10] A certain analogy to a state of equilibrium is presented by the situation when X_t and Y_t are periodic functions with a constant amplitude. Then the state of the system, it is true, is not invariant but is repeated at given intervals of time. Such a situation can be described as *cyclic equilibrium.* At given intervals of time the system returns to the same state.

[11] Since $X_t = \text{const.}$, the subscript t becomes superfluous.

state of the system in equilibrium. A condition for a system's state of equilibrium to be possible, therefore, is the existence of a vector solving equation (8.1). In order for such a solving vector to exist, the transformation R in equation (8.1) must have certain properties, namely for at least one vector $\boldsymbol{X}$ the right-hand side of equation (8.1) must be equal to $\boldsymbol{X}$.

Since in (8.1a) and (8.1b) the variable τ becomes eliminated by summation or integration, the right-hand side of the expression is a function solely of the vector $\boldsymbol{X}$. Let us denote this function by $F(\boldsymbol{X})$. Equation (8.1), therefore, can be written in the form[12]

$$\boldsymbol{X} = F(\boldsymbol{X}). \tag{8.2}$$

The geometric interpretation of this equation is as follows. The vector $\boldsymbol{X}$ has $\sum_{r=1}^{N} m_r$ components. The geometric picture of the right-hand side of the equation, i.e. of the function $F(\boldsymbol{X})$, therefore, constitutes a $\sum_{r=1}^{N} m_r$-dimensional hypersurface. On the other hand, the geometrical picture of the left-hand side of the equation is a $\sum_{r=1}^{N} m_r$-dimensional hyperplane inclined at 45° to the coordinate axes. The points of intersection or tangency of the hypersurface and hyperplane determine the solution (or solutions) of the equation. This is illustrated symbolically by the two-dimensional diagram below:

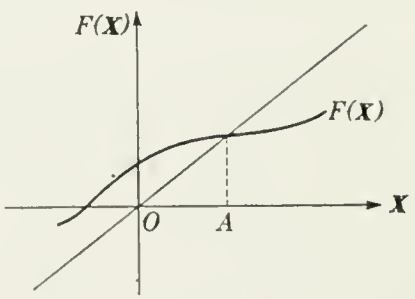

FIG. 9

[12] For the sake of uniform notation, we can also write (8.1c) in this way.

In this diagram, OA is the solution of the equation. In order for equation (8.2) or the identical equation (8.1) to have a solution the hypersurface representing the function $F(X)$ must have points in common with the hyperplane. It must intersect it or at least have points of tangency with it.

A similar geometrical interpretation is obtained if the equilibrium equation is considered in the form $Y = F(Y)$.

Not all systems, therefore, can have a state of equilibrium. Equilibrium is possible only in systems in which the transformation R (or the transformation P), expressing the mode of action of the system, has the properties ensuring the existence of a solution of the equilibrium equation.

If the system can be in equilibrium and if the input and output states of its elements are such that the vector X or the vector Y which express them is a solution of the equilibrium equation, the system remains in the state of equilibrium. With other input and output states, the system is not in a state of equilibrium. Its state changes with time. However, this change may tend to a state of equilibrium. In this case, we say that the system is *stable*. A stable system, even if not in a state of equilibrium at the outset, tends to equilibrium with time.

The mathematical expression for the stability of a system is

$$\lim_{t\to\infty} X_t = \hat{X} \tag{8.3}$$

or

$$\lim_{t\to\infty} Y_t = \hat{Y}, \tag{8.4}$$

where $\hat{X}$ and $\hat{Y}$ are solutions of the equilibrium equation. Since, as we know, the shapes of the vector functions of time X_t and Y_t depend on the value of those functions at certain, or all, moments in the initial interval of time $[0, \theta]$, the condition expressed by (8.3) or (8.4) may be satisfied for certain initial values of that function and not be satisfied for other initial values. The set of initial values for which this condition is satisfied shall be called the system's *domain of stability*. The system may be stable in one domain of initial values of its states and may not be stable in another domain of initial values.

By

$$\Delta X_t = X_t - \hat{X}_t \tag{8.5}$$

and

$$\Delta Y_t = Y_t - \hat{Y}_t \tag{8.6}$$

we denote the deviation of the input or output states of the system's elements from the respective equilibrium states. The stability of the system can then be expressed in the form

$$\lim_{t\to\infty} \Delta X_t = 0 \tag{8.7}$$

or

$$\lim_{t\to\infty} \Delta Y_t = 0. \tag{8.8}$$

Let ΔX_t be a small deviation from the state of equilibrium. Let us also assume that the transformations R_{rs} defined above in (6.4) have first derivatives with respect to the vectors $x_t^{(r)}$ $(r = 1, 2, \ldots, N)$.[13] Then, from the law of motion of the systems (7.5) or (7.17), we obtain the following. In the case of sudden action:

$$\Delta X_{t+\theta} = \left[\frac{\partial R}{\partial X_t}\right]_{X_t=\hat{X}} \Delta X_t. \tag{8.9a}$$

And in the case of a gradual action:

$$\Delta X_t = \sum_{\tau=0}^{\theta} \left[\frac{\partial R\,(\tau)}{\partial X_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}} \Delta X_{t-\tau} \tag{8.9b}$$

or

$$\Delta X_t = \int_0^{\theta} \left[\frac{\partial R\,(\tau)}{\partial X_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}} \Delta X_{t-\tau}\,d\tau. \tag{8.9c}$$

The symbol

$$\left[\frac{\partial R}{\partial X_t}\right]_{X_t=\hat{X}} \quad \text{or} \quad \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}}$$

denotes a functional (Jacobian) matrix the elements of which assume values corresponding to the state of equilibrium.

[13] Our argument could be extended to cover non-differentiable transformations as well, but their presentation then becomes more complicated. For this reason we limit ourselves here to differentiable transformations.

Therefore,

$$\left[\frac{\partial R}{\partial X_t}\right]_{X_t=\hat{X}} = \left[\frac{\partial R_{rs}}{\partial x_t^{(r)}}\right]_{x_t^{(r)}=\hat{x}^{(r)}} \tag{8.10a}$$

and

$$\left[\frac{\partial R(\tau)}{\partial X_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}} = \left[\frac{\partial R_{rs}(\tau)}{\partial x_{t-\tau}^{(r)}}\right]_{x_{t-\tau}^{(r)}=\hat{x}^{(r)}} \tag{8.10b}$$

where $\hat{x}^{(r)}$ is the value of the vector $x_t^{(r)}$ when the system is in a state of equilibrium. As can be seen, the elements of the matrix (8.10a) are constants whereas the elements of the matrix (8.10b) are functions of the parameter τ. The subscripts of the matrices (8.10a) and (8.10b) are transposed as compared with the subscripts of the matrix (6.18).[14]

The expressions (8.9a) and (8.9b) are vectorial difference equations and the expression (8.9c) is a vectorial integral equation. The solution of these equations yields a function of time

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[14] If the compound vector ΔX_t is interpreted as a quasi-diagonal matrix in accordance with formula (6.9a), we obtain from the right-hand side of (6.11) the following matrix equation

$$\begin{bmatrix} 0 & \Delta x_{t+\theta}^{(1)} & \dots & \Delta x_{t+\theta}^{(1)} \\ \Delta x_{t+\theta}^{(2)} & 0 & \dots & \Delta x_{t+\theta}^{(2)} \\ \dots & \dots & \dots & \dots \\ \Delta x_{t+\theta}^{(N)} & \Delta x_{t+\theta}^{(N)} & \dots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{\partial R_{21}}{\partial x_t^{(2)}} \Delta x_t^{(2)} & \dots & \frac{\partial R_{N1}}{\partial x_t^{(N)}} \Delta x_t^{(N)} \\ \frac{\partial R_{12}}{\partial x_t^{(1)}} \Delta x_t^{(1)} & 0 & \dots & \frac{\partial R_{N2}}{\partial x_t^{(N)}} \Delta x_t^{(N)} \\ \dots & \dots & \dots & \dots \\ \frac{\partial R_{1N}}{\partial x_t^{(1)}} \Delta x_t^{(1)} & \frac{\partial R_{2N}}{\partial x_t^{(2)}} \Delta x_t^{(2)} & \dots & 0 \end{bmatrix}.$$

The left-hand side of this equation is equivalent to the compound vector $\Delta X_{t+\theta}$. In this way we obtain the vector equation (8.9a). The vector equations (8.9a) or (8.9b) are obtained in an analogous way. In the matrix equation presented here the subscripts are transposed, for otherwise it would be necessary to write $\Delta X_t \left[\frac{\partial R}{\partial \Delta X_t}\right]$, which would be inconvenient.

$\Delta \boldsymbol{X}_t$ which represents the behaviour in time of the deviation from the system's state of equilibrium.

As we know, the solution is a funetion of the type[15]

$$\Delta \boldsymbol{X}_t = \sum_j \boldsymbol{K}_j \lambda_j^t , \tag{8.11}$$

where $\boldsymbol{K}_j$ are veetors eomposed of eonstants, and λ_j are the roots of the eharaeteristie equation of the differenee or integral equation. At the moment $t = 0$ we have

$$\Delta \boldsymbol{X}_0 = \sum_j \boldsymbol{K}_j ; \tag{8.12}$$

in others words, $\sum_j \boldsymbol{K}_j$ stands for the deviation of the input states from the equilibrium states at the initial moment. In the ease of the integral equation (8.9c) there is always $\lambda_j \geqslant 0$ so that we ean write $\lambda_j = \mathrm{e}^{\varkappa_j}$ and

$$\Delta \boldsymbol{X}_t = \sum_j \boldsymbol{K}_j \mathrm{e}^{\varkappa_j t} . \tag{8.13}$$

The system is stable if

$$\sum_{t \to \infty} \Delta \boldsymbol{X}_t = \lim_{t \to \infty} \sum_j \boldsymbol{K}_j \lambda_j^t = 0. \tag{8.14}$$

This requires that $|\lambda_j| < 1$ (or in the case of the integral equation $\varkappa_j < 0$) for all j. If the root λ_j is a real number the signifieanee of this condition is obvious. On the other hand, if λ_j is a eomplex number, then we write

$$\lambda_j = \alpha_j + \beta_j \mathrm{i} = \varrho_j (\cos \omega_j + \mathrm{i} \sin \omega_j)$$

and then

$$|\lambda_j| = \varrho_j = \sqrt{\alpha_j^2 + \beta_j^2} . \tag{8.15}$$

The corresponding element of the sum (8.11) then has the form

$$\boldsymbol{K}_j \varrho_j^t (\cos \omega_j t + \mathrm{i} \sin \omega_j t), \tag{8.16}$$

[15] If λ_j is a multiple root, then instead of $\boldsymbol{K}_j$ there appears a polynomial in $\Delta \boldsymbol{X}_t$, the degree of the polynomial being one less than the multiplicity of the root λ_j. This does not change our further conclusions.

from which it is immediately seen that the condition of the stability of the system is $\varrho_j < 1$.

In order to investigate the conditions for the stability of the system it is necessary, therefore, to investigate the characteristic equation of equations (8.9) and to determine the nature of its roots. The characteristic equation is obtained by substituting the expression $\boldsymbol{K}\lambda^t$ for $\Delta \boldsymbol{X}_t$ in the equations (8.9).

For the difference equation (8.9a) we then obtain

$$\boldsymbol{K}\lambda^{t+\theta} = \left[\frac{\partial R}{\partial \boldsymbol{X}_t}\right]_{X_t=\hat{X}} \boldsymbol{K}\lambda^t.$$

This can be satisfied for a non-zero vector $\boldsymbol{K}$ only if the determinant

$$\left| \left[\frac{\partial R}{\partial \boldsymbol{X}_t}\right]_{X_t=\hat{X}} \lambda^{-\theta} - I \right| = 0, \tag{8.17a}$$

where I is the unit (identity) matrix. This is the characteristic equation of the difference equation (8.9a).

In a similar way we find for the difference equation (8.9b)

$$\boldsymbol{K}\lambda^t = \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial \boldsymbol{X}_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}} \boldsymbol{K}\lambda^{t-\tau}.$$

From this we obtain the characteristic equation

$$\left| \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial \boldsymbol{X}_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}} \lambda^{-\tau} - I \right| = 0. \tag{8.17b}$$

And for the integral equation (8.9c) we have

$$\boldsymbol{K}\lambda^t = \int_0^{\theta} \left[\frac{\partial R(\tau)}{\partial \boldsymbol{X}_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}} \boldsymbol{K}\lambda^{t-\tau} \mathrm{d}\tau.$$

Hence the characteristic equation

$$\left| \int_0^{\theta} \left[\frac{\partial R(\tau)}{\partial \boldsymbol{X}_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}} \lambda^{-\tau} \mathrm{d}\tau - I \right| = 0. \tag{8.17c}$$

From the characteristic equation it follows that in order for the stability condition $|\lambda_j| < 1$ to be satisfied, the system

must possess at least one feed-back coupling. Moreover, the feed-back couplings in the system must have certain specific properties.

As we know, the transformations (7.4) are defined only for $s \neq r$; for by definition no element of the system is coupled with itself. For that reason in the matrix R of operators of these transformations the diagonal elements are zero (cf. (6.15) above). In view of that, the derivative expressions $\partial R_{rs}/\partial x^{(r)}$ appearing in the diagonal of the matrix (8.10) are also zero. This is true for all systems. If, however, a system does not contain feed-back couplings then, as follows from (6.4) and (4.7), and taking account of the transposition of the subscripts in the matrix (8.10), all the elements of the matrix above the diagonal are also zero.[16] In the case of a system without feed-back couplings the matrix (8.10a), therefore, is of the form

$$\left[\frac{\partial R}{\partial X_t}\right]_{X_t=\hat{X}} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ & 0 & 0 & \dots & 0 \\ & & 0 & \dots & 0 \\ & & & \cdots & \\ & \text{non-zero} & & & \\ & \text{elements} & & & 0 \end{bmatrix}. \tag{8.18}$$

Of such a form are also all the principal minors of this matrix; the matrix, therefore, is of zero order. The same is true of the matrix (8.10b).

Multiplying the characteristic equation (8.17) by λ^N, we obtain on the left-hand side a determinant in which all the elements on the diagonal are equal to $-\lambda$. If the system does not possess feed-back couplings, the determination has only zeros above the diagonal; its value, therefore, is $(-\lambda)^N$. The characteristic equation then reduces to

$$(-\lambda)^N = 0.$$

[16] In the matrix (6.18) as well as in the structure matrix of the system (4.5), in the absence of feed-back couplings all the elements below the diagonal are zero. For the reason given in footnote 14, on p. 43, in the matrix (8.10) the subscripts are transposed. Therefore, in this matrix in the absence of feed-back couplings all the elements above the diagonal are zero.

From this we have

$$\lambda = 0. \tag{8.19}$$

In this case (cf. (8.12) above) we have

$$\Delta \boldsymbol{X}_0 = \sum_j \boldsymbol{K}_j 0^0 \ ;$$

hence the $\boldsymbol{K}_j$ are indeterminate. Therefore, the expression

$$\Delta \boldsymbol{X}_t = \sum_j \boldsymbol{K}_j \lambda_j^t \ ,$$

is also indeterminate, that is, the function describing the behaviour in time of the deviation from the equilibrium state.

Thus in the case when the system has no feed-back couplings, any state of the system other than the equilibrium state is indeterminate. Once thrown out of equilibrium, the system can assume any arbitrary state. A system of this kind is said to be *neutral.* It is "dead", as it were; thrown out of equilibrium, it remains in any arbitrary state in which it happens to be.

From the equivalence of the differential or integral equations (7.8) and (7.18), and (7.5) and (7.17), it follows that a system without feed-back couplings is neutral with respect to both the output and input states of its elements. For the output states we find that the deviation from the equilibrium state $\Delta \boldsymbol{Y}_t$ in such cases assumes arbitrary values.

The existence of at least one feed-back coupling is thus a necessary condition for the stability of a system. But it is not a sufficient condition. In order for the system to be stable, the feed-back couplings (or coupling if there is only one) must possess certain specific properties. The feed-back couplings must be such that the roots of the characteristic equation all have absolute values smaller than unity, i.e. $|\lambda_j| < 1$ for all λ_j. The feed-back couplings at which this occurs are called *compensatory feed-back couplings.*

In order to understand the nature of compensatory feed-back couplings, let us consider a simple example. Let the system consist of only two elements and let its action in time be sudden;

let us also assume that $\theta = 1$. Then the characteristic equation (8.17a) assumes the form

$$\begin{vmatrix} -1 & \dfrac{\partial R_{21}}{\partial x_t^{(2)}}\lambda^{-1} \\ \dfrac{\partial R_{12}}{\partial x_t^{(1)}}\lambda^{-1} & -1 \end{vmatrix} = 0 \quad \text{for } x_t^{(1)} = \hat{x}^{(1)} \text{ and } x_t^{(2)} = \hat{x}^{(2)}. \tag{8.19'}$$

Multiplying by λ^2 we can write it in the form

$$\begin{vmatrix} -\lambda & \dfrac{\partial R_{21}}{\partial x_t^{(2)}} \\ \dfrac{\partial R_{12}}{\partial x_t^{(1)}} & -\lambda \end{vmatrix} = 0 \quad \text{for } x_t^{(1)} = \hat{x}^{(1)} \text{ and } x_t^{(2)} = \hat{x}^{(2)}.$$

If the system has no feed-back coupling, then $\partial R_{21}/\partial x_t^{(2)} = 0$ and the root of the characteristic equation is $\lambda = 0$. The system is then neutral. On the other hand, if there is a feed-back coupling, the characteristic equation has two roots, viz.

$$\lambda_1 = \sqrt{\frac{\partial R_{12}}{\partial x_t^{(1)}}\frac{\partial R_{21}}{\partial x_t^{(2)}}} \quad \text{and} \quad \lambda_2 = -\sqrt{\frac{\partial R_{12}}{\partial x_t^{(1)}}\frac{\partial R_{21}}{\partial x_t^{(2)}}} \tag{8.20}$$

$$(x_t^{(1)} = \hat{x}^{(1)},\ x_t^{(2)} = \hat{x}^{(2)}).$$

These roots are real if the sign of $\partial R_{12}/\partial x_t^{(1)}$ is the same as that of $\partial R_{21}/\partial x_t^{(2)}$, and they are imaginary (and conjugate) if the signs are opposite. If the signs are the same we say that the feed-back coupling is *positive*; otherwise we say that the feed-back coupling is *negative*.[17]

In the case of a positive feed-back coupling an increase (decrease) in the value of the variable representing the input state of a certain element in the system increases (decreases) the value of the variable representing the input state of the element feed-back coupled with it. In the case of a negative feed-back coupling

[17] Cf. Greniewski, *Cybernetics without Mathematics*, pp. 37–41.

an increase (decrease) in the first-mentioned value decreases (increases) the other value.

The function representing the behaviour in time of the system's deviation from the state of equilibrium is now of the form

$$\Delta \boldsymbol{X}_t = \boldsymbol{K}_1\lambda_1^t + \boldsymbol{K}_2\lambda_2^t.$$

As can be seen from (8.20), we have $\lambda_2 = -\lambda_1$ and, therefore, by writing $\boldsymbol{K} = \boldsymbol{K}_1 - \boldsymbol{K}_2$, we obtain

$$\Delta \boldsymbol{X}_t = \boldsymbol{K}\lambda^t. \tag{8.21}$$

If the feed-back coupling is positive, λ is real and $|\Delta \boldsymbol{X}_t|$ is a function monotonically increasing or decreasing, according to whether $|\lambda| > 1$ or $|\lambda| < 1$. If, on the other hand, the feed-back coupling is negative, λ is imaginary and we ean write

$$\Delta \boldsymbol{X}_t = K\varrho^t(\cos\ \omega t + \mathrm{i}\sin\ \omega t). \tag{8.22}$$

The function is then oscillatory and the oscillations increase or decrease according to whether $\varrho = |\lambda| < 1$ or > 1.

We see, therefore, that in our example a positive feed-back coupling produces a monotonic change in the deviation from the equilibrium state whereas a negative feed-back coupling produces changes of an oscillatory character.

The system is stable if $|\lambda| < 1$. In our example this reduces to

$$\frac{\partial R_{12}}{\partial x_t^{(1)}}\,\frac{\partial R_{21}}{\partial x_t^{(2)}} < 1 \qquad (x_t^{(1)} = \hat{x}^{(1)},\ x_t^{(2)} = \hat{x}^{(2)}). \tag{8.23}$$

This is both a necessary and sufficient condition. It can be written in the form:[18]

$$\left|\frac{\partial R_{21}}{\partial x_t^{(2)}}\right| < \frac{1}{\left|\dfrac{\partial R_{12}}{\partial x_t^{(1)}}\right|} \qquad (x_t^{(1)} = \hat{x}^{(1)},\ x_t^{(2)} = \hat{x}^{(2)}). \tag{8.24}$$

[18] In this form the condition for the system is stability is analogous to the well-known Nyquist criterion of stability employed in industrial automation. For the Nyquist criterion, see Werner zur Megede, *Einfuehrung in die Technik Selbsttaetiger Regelungen*, de Gruyter, Berlin, 1956, pp. 128–129; see also V. Broïda, *Automatisme, régulation automatique, servomécanismes*, Dunod, Paris, 1957, pp. 297–299. The Nyquist criterion concerns automatic regulation based on oscillation frequency. The condition (8.24) constitutes a generalization of the Nyquist criterion.

This means that in order that the system be stable in the neighbourhood of the equilibrium state the absolute value of the reaction of the input of the feed-baek coupled elemcnt E_1 to a ehange of the input of element E_2 must be lcss than the reeiproeal of the absolute valuę of the reaetion of the element E_2 to a change in the input of element E_1. In other words, in the event of a deviation of the system from equilibrium, the feedbaek eoupling must weaken the ehanges in the inputs of the elements of the system.[19]

This is the eompensatory effect of the feed-baek eoupling, required for the stability of the system.

It should be noted that the stability of the system does not require the feed-baek eoupling to be negative. A positivc feedbaek eoupling, too, may havc a eompensatory effeet if the eondition (8.23) or (8.24)[20] is satisfied. In the ease of a negative feed-baek eoupling a stable system tends to the state of equilibrium by way of deereasing (damped) oscillations; in the ease of a positive feed-baek eoupling, a stable system tends to the equilibrium state by way of a monotonic approaeh to it.[21] On

[19] Here we have an analogy with the Le Chatelier principle which appears in thermodynamics. This principle states that in a stable system an external action throwing the system out of equilibrium "causes in it processes tending to reduce the results of that action" (L. Landau and E. Lifshitz, *Statistical Physics*. Cf. English translation by Pergamon Press, London, 1960, p. 82). Condition (8.24) can also be regarded as a generalization of the Le Chatelier principle.

[20] Attention to this has been drawn by W. Ross Ashby, *Introduction to Cybernetics*, Chapman and Hall, London, 1956, p. 98. The wide-spread erroneous view that there must be negative feed-back coupling in order for the system to be stable is due to the fact that such couplings are employed as a rule in industrial automation technique. In this technique, equilibrium is attained by way of damped oscillations.

[21] We disregard here the short-period oscillations with period equal to two units of time which occur if $\lambda < 0$ and if t assumes only integer values. In this case $\lambda^t > 0$ for even values of t and $\lambda^t < 0$ for odd values of t. Such oscillations corresponding to various negative values of the roots λ_j are superimposed upon the monotonic path leading to the state of equilibrium. This problem does not appear in the integral equation for in this case $\lambda > 0$ and t assumes all real values. It occurs only in discrete processes.

the other hand, a negative feed-back coupling does not, in itself, ensure the stability of the system. If condition (8.23) is not satisfied, the feed-back coupling causes oscillation with an amplitude constant if $|\lambda| = 1$, or increasing if $|\lambda| > 1$.

Now we proceed to the more general case when the system contains N elements, its action still being sudden in time, and $\theta = 0$. The characteristic equation then is of the form (8.17a). Multiplying it by λ^N, we obtain on the left-hand side a determinant in which the expressions on the diagonal are all equal to $-\lambda$. Writing $\lambda = \dfrac{\nu+1}{\nu-1}$ we find that the condition for the stability of the system can be formulated in the form of a condition concerning ν. If λ is a real number, this condition is simply $\nu < 0$. If, however, λ is a complex number, this condition is $\operatorname{Re} \nu < 0$, where $\operatorname{Re} \nu$ denotes the real part of the complex number ν.[22]

Several criteria are known for determining whether the real part of the root of the characteristic equation is negative. Treating ν as the unknown of the characteristic equation, we can decide by means of these criteria whether $\nu < 0$ (or $\operatorname{Re} \nu < 0$). If this is the case, the condition for the stability of the system is satisfied. This condition is both necessary and sufficient.

If we write $\lambda = \dfrac{\nu+1}{\nu-1}$, the determinant which constitutes the left-hand side of the characteristic equation has the expression $-\dfrac{\nu+1}{\nu-1}$ everywhere in its diagonal. If $\left(\dfrac{1}{\nu-1}\right)^N$ is bracketed out

[22] If λ is a real number, then from $\lambda = (\nu+1)/(\nu-1)$ it follows directly that $|\lambda| < 1$ if and only if $\nu < 0$. If, on the other hand, λ is a complex number, we write $\lambda = \alpha+\beta\,\mathrm{i}$ and $\nu = a+b\mathrm{i}$. Then

$$\lambda = \alpha+\beta\mathrm{i} = \frac{(a+1)+b\mathrm{i}}{(a-1)+b\mathrm{i}}.$$

We have $|\lambda|^2 = \alpha^2+\beta^2 = \dfrac{(a+1)+b\mathrm{i}}{(a-1)+b\mathrm{i}}\,\dfrac{(a+1)-b\mathrm{i}}{(a-1)-b\mathrm{i}} = \dfrac{(a+1)^2+b^2}{(a-1)^2-b^2}$. From this it follows directly that $|\lambda|^2 < 1$ and, therefore, $|\lambda| < 1$, too, if and only if $\alpha = \operatorname{Re} \nu < 0$.

of the determinant, the expression $(-1-\nu)$ remains in the diagonal. Two cases call for particular attention.

One is the case when the determinant is symmetric. Then, as we know, the roots $\nu_1, \nu_2, \ldots, \nu_N$ of the characteristic equation are all real. They are negative if the matrix of the characteristic equation is negative definite. The matrix of the characteristic equation is negative definite if

$$-1 < 0, \quad \begin{vmatrix} -1 & \dfrac{\partial R_{21}}{\partial x_t^{(2)}} \\ \dfrac{\partial R_{12}}{\partial x_t^{(1)}} & -1 \end{vmatrix} > 0,$$

$$\begin{vmatrix} -1 & \dfrac{\partial R_{21}}{\partial x_t^{(2)}} & \dfrac{\partial R_{31}}{\partial x_t^{(3)}} \\ \dfrac{\partial R_{12}}{\partial x_t^{(1)}} & -1 & \dfrac{\partial R_{32}}{\partial x_t^{(3)}} \\ \dfrac{\partial R_{13}}{\partial x_t^{(1)}} & \dfrac{\partial R_{23}}{\partial x_t^{(2)}} & -1 \end{vmatrix} < 0, \quad \text{etc.} \tag{8.25}$$

for $x_t^{(1)} = \hat{x}^{(1)}$, $x_t^{(2)} = \hat{x}^{(2)}$, $x_t^{(3)} = \hat{x}^{(3)}$.

As can be seen, the second of these inequalities is identical with (8.23) or (8.24). The further inequalities of the above sequence may be regarded as a generalization of the second, account being taken of the couplings of an increasing number of elements in the system.

In the case when the matrix of the characteristic equation is symmetric all feed-back couplings are positive. The symmetry of the matrix means that the positive feed-back couplings are symmetric with respect to the corresponding direct couplings. The magnitude of the reaction of the input of the element E_r to a change in the input of the element E_s is then equal to the magnitude of the reaction of the input of the element E_s to a change in the input of the element E_r. The opposite occurs when the absolute values of these reactions are equal but the relations are of opposite sign. Then the feed-back couplings are negative and skew symmetric with respect to the corresponding direct

couplings. The matrix of the characteristic equation is then skew symmetric. This is the second special case which we shall consider.

If the matrix of the characteristic equation is skew symmetric then, as is well known, all the roots of the equation are imaginary, i.e. $\operatorname{Re} \nu_j = 0$. The function ΔX_t representing the deviation from the system's state of equilibrium is then oscillatory. Since $\operatorname{Re} \nu_j = 0$, we have $|\lambda_j| = \varrho_j = 1$. In consequence, the function ΔX_t assumes the form

$$\Delta X_t = \sum_j K_j (\cos \omega t + \mathrm{i} \sin \omega t).$$

The oscillations thus have a constant amplitude. The feed-back couplings, being skew symmetric, fully compensate the action of the corresponding direct couplings. This keeps the system in constant oscillatory motion.

Now let us consider the quite general case when the feed-back couplings in the system may be either negative or positive and need not be symmetric. We still assume that action is sudden in time, and for the sake of simplicity, we assume that $\theta = 1$. Then the vectorial difference equation (8.9a) is

$$\Delta X_{t+1} = \left[\frac{\partial R}{\partial X_t}\right]_{X_t = \hat{X}} \Delta X_t \,.$$

According to (6.9a), the compound vectors ΔX_t and ΔX_{t+1} can be interpreted as quasi-diagonal matrices. Let us premultiply both sides of the vectorial difference equation by the transpose of the matrix on each side (the transpose of a matrix we denote by the addition to the matrix of the symbol "′"). Then we obtain

$$\Delta X'_{t+1} \Delta X_{t+1} = \left\{\left[\frac{\partial R}{\partial X_t}\right]_{X_t = \hat{X}} \Delta X_t\right\}' \left[\frac{\partial R}{\partial X_t}\right]_{X_t = \hat{X}} \Delta X_t \,,$$

which can be written in the form

$$\Delta X'_{t+1} \Delta X_{t+1} = \Delta X'_t \left[\frac{\partial R}{\partial X_t}\right]' \left[\frac{\partial R}{\partial X_t}\right]_{X_t = \hat{X}} \Delta X_t \,. \qquad (8.26a)$$

Putting $\Delta X_t = K\lambda^t$ and, in consequence, $\Delta X'_t = K'\lambda^t$, we get

$$K'K\lambda^{2(t+1)} = K' \left[\frac{\partial R}{\partial X_t}\right]' \left[\frac{\partial R}{\partial X_t}\right]_{X_t = \hat{X}} K\lambda^{2t}.$$

Thus the characteristic equation is

$$\left| \left[\frac{\partial R}{\partial X_t}\right]' \left[\frac{\partial R}{\partial X_t}\right]_{X_t=\hat{X}} \lambda^{-2} - I \right| = 0. \tag{8.27a}$$

The product

$$\left[\frac{\partial R}{\partial X_t}\right]' \left[\frac{\partial R}{\partial X_t}\right]_{X_t=\hat{X}}, \tag{8.28a}$$

known in algebra as the Gaussian transformation, is a symmetric matrix. In consequence, the matrix of the characteristic equation is also symmetric. Writing $\lambda^2 = \dfrac{\nu+1}{\nu-1}$ and treating ν as the unknown of the characteristic equation, we find that the roots $\nu_1, \nu_2, \ldots, \nu_N$ are real. These roots are negative if the matrix of the characteristic equation, viz.

$$\left[\frac{\partial R}{\partial X_t}\right]' \left[\frac{\partial R}{\partial X_t}\right]_{X_t=\hat{X}} - I \tag{8.29a}$$

is negative definite.[23] Then, for all j, $\lambda_j^2 < 1$, and hence $|\lambda_j| < 1$. If $\lambda_j^2 < 0$, then λ_j is imaginary; the function $\Delta \boldsymbol{X}_t$ is oscillatory and the oscillations are diminishing (damped).

There is a striking analogy in form between this condition and (8.23) or (8.25). This condition is a generalization of the condition (8.23) (for a system consisting of two elements) and of the condition (8.25) (for a system of N elements) obtaining when all couplings are positive and symmetric. The stability condition (8.29a) holds for any system with negative as well as positive and not necessarily symmetric couplings. This is the

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[23] This condition is mentioned by P. A. Samuelson in *Foundations of Economic Analysis*, Harvard, Cambridge, 1947, p. 438. An essentially similar condition has been formulated by S. Karlin in *Mathematical Methods and Theory in Games, Programming and Economics*, Pergamon Press, London, 1959, vol. I, p. 320. Karlin's condition requires the existence of such a set of positive numbers $c_1, c_2, \ldots, c_N$ that (employing our notation) $\sum_{r \neq s} c_r \dfrac{\partial R_{rs}}{\partial x_t^{(r)}} < 1$ for $r, s = 1, 2, \ldots, N$ and $x_t^{(r)} = \hat{x}^{(r)}$.

general condition for the compensatory character of the structure of couplings in a system in which the action in time is sudden.

A more complex condition for the stability of a system is obtained if the action is gradual in time but discrete. Then we premultiply the vectorial difference equation (8.9b) by the transpose of the matrix appearing on each side. We obtain

$$\begin{aligned}
&\Delta X_t' \Delta X_t \\
&= \left\{ \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}} \right]_{X_{t-\tau}=\hat{X}} \Delta X_{t-\tau} \right\}' \left\{ \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}} \right]_{X_{t-\tau}=\hat{X}} \Delta X_{t-\tau} \right\} \\
&= \sum_{\tau=0}^{\theta} \Delta X_{t-\tau}' \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}} \right]'_{X_{t-\tau}=\hat{X}} \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}} \right]_{X_{t-\tau}=\hat{X}} \Delta X_{t-\tau}. \qquad (8.26b)
\end{aligned}$$

Writing $\Delta X_t = K\lambda^t$, we have

$$\begin{aligned}
&K' K \lambda^{2t} \\
&= K' \sum_{\tau=0}^{\theta} \lambda^{t-\tau} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}} \right]_{X_{t-\tau}=\hat{X}} \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}} \right]_{X_{t-\tau}=\hat{X}} K \lambda^{t-\tau}. \qquad (8.27b)
\end{aligned}$$

The characteristic equation, therefore, is

$$\left| \left\{ \sum_{\tau=0}^{\theta} \lambda^{-\tau} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}} \right]' \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}} \right] \lambda^{-\tau} \right\}_{X_{t-\tau}=\hat{X}} - I \right| = 0. \qquad (8.28b)$$

In this equation the product of the two sums is a symmetric matrix (Gaussian transformation) and in view of this, the matrix of the characteristic equation is also symmetric. Writing $\lambda = (\nu+1)/(\nu-1)$ and considering ν as the unknown of the characteristic equation, we find the roots of this equation. The unknowns ν appear in the matrix of the characteristic equation in the form of polynomials of degree 2θ. Their roots, therefore, are not necessarily real, they can be complex as well. The sign of their real parts depends on the properties of the matrix of the characteristic equation, i.e. of the matrix

$$\sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}} \right]' \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}} \right]_{X_{t-\tau}=\hat{X}} - I. \qquad (8.29b)$$

Performing similar operations on the vectorial integral equation (8.9c) we find the condition for the stability of the system in the case when the gradual action in time is continuous. The condition depends on the properties of the symmetric matrix

$$\int_0^\theta \left[\frac{\partial R(\tau)}{\partial \boldsymbol{X}_{t-\tau}}\right]' \mathrm{d}\tau \int_0^\theta \left[\frac{\partial \hat{R}(\tau)}{\partial \boldsymbol{X}_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}} \mathrm{d}\tau - I\,. \tag{8.29c}$$

Condition (8.29) is the most general form of the condition for the stability of a system. It can be also formulated similarly with respect to ΔY_t, i.e. to the deviation of outputs from the equilibrium state.

The final result of our considerations is as follows. A stable system must possess at least one feed-back coupling and the feed-back couplings of a stable system must be compensatory. The most general condition for the compensatory character of feed-back couplings is that the matrix (8.29) be negative definite. All other conditions can be regarded as particular cases of this general condition.

The compensatory feed-back couplings in a stable system we call the *regulators* or *stabilizers* of the system. The process of a stable system tending to a state of equilibrium is called *self-regulation* of the system.[24]

The mathematical expression of the stability of a system, is, as we have seen, the condition $|\lambda_j| < 1$ for all roots of the characteristic equation. If the condition (8.29) is not satisfied, we have $|\lambda_j| = 1$ or $|\lambda_j| > 1$. If all $|\lambda_j| = 1$ then, as can be seen from (8.11), we have

$$\Delta \boldsymbol{X}_t = \sum_j \boldsymbol{K}_j \quad \text{for} \quad \lambda_j \text{ real}$$

or

$$\Delta \boldsymbol{X}_t = \sum_j \boldsymbol{K}_j (\cos \omega_j t + \mathrm{i} \sin \omega_j t) \quad \text{for} \quad \lambda_j \text{ complex.}$$

[24] In biology, self-regulation of the organism (e.g. maintenance of constant temperature in the human body) is usually called *homeostatis*.

The system, therefore, either remains in any initial state in which it happens to be or oscillates around this state with a constant amplitude. A system of this sort is said to be *quasi-stable.*[25]

On the other hand, if there is at least one root for which $|\lambda_j| > 1$, we have

$$\lim_{t\to\infty} \Delta X_t = \lim_{t\to\infty} \sum_j K_j \lambda_j^t = \infty \quad \text{or} \quad -\infty .$$

Such a system shall be called *unstable*; thrown out of equilibrium, it departs further and further from that state, without end. As we know, the instability of a system is connected with the existence of feed-back couplings with certain properties (in the absence of feed-back couplings, the system is neutral). The feed-back couplings causing the instability of the system are called *cumulative feed-back couplings* or, more simply, the *cumulators* of the system. The process of the development of an unstable system thrown out of equilibrium shall be called a *cumulative process.*

As we know (see p. 44) there exists a specified domain of stability of the systems or, respectively, a specified domain of instability or quasi-stability. Investigations of the conditions for the stability of a system, therefore, require that the domain of stability be determined. For our purposes this problem need not be considered.

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[25] A quasi-stable system can be regarded as being stable in a wider sense. In such a system every state can be regarded as a state of equilibrium or possibly as a state of cyclical equilibrium if the system oscillates with a constant amplitude. Cf. footnote 9 on p. 33.

IX. ERGODIC PROCESSES SELF-STEERING

THE PROCESS of a stable system tending to an equilibrium state is a special case of a wider category of development processes which we call *ergodic*. Ergodic processes are such development processes which, with time, become independent of the initial state of the system.[26] In a stable system the development tends to a certain state independent of the initial state of the system, namely to the state of equilibrium. The stability of the system lies precisely in this. In the general case of ergodic processes, the development of the system tends to conformity with a given law of development independent of the initial state of the system.

Let us denote by $\hat{X}_t$ and $\hat{Y}_t$ vector functions of time independent of the initial state of the system. These functions express a certain law of development for the states of the inputs and outputs of the system. Let X_t and Y_t be, as before, the vector functions of time constituting a solution of the law of development of the system, i.e. solutions of the vectorial difference or integral equations (7.5), (7.8) or (7.17). These functions represent the actual development of the system in time which, as we know, depends on the state of the system at certain initial moments. The development of the system is an ergodic process if

$$\lim_{t\to\infty} X_t = \hat{X}_t \tag{9.1}$$

and

$$\lim_{t\to\infty} Y_t = \hat{Y}_t . \tag{9.2}$$

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[26] The term *ergodic* is used here in a way analogous to that in the theory of Markov chains. A Markov chain is said to be ergodic when the probability distribution in successive trials converges to a limiting distribution independent from the initial distribution. Cf. for instance, L. Takács, *Stochastic Processes*, Methuen and Co., Ltd., London 1960, pp. 18 and 23.

In view of (6.17) these expressions are equivalent; one of them implies the other.

It is easily seen that the expression for the stability of a system is a special case of the expression given for an ergodic process. This special case occurs when $\hat{X}_t = \hat{X} = \text{const.}$ and $\hat{Y}_t = \hat{Y} = \text{const.}$

Functions $\hat{X}_t$ or $\hat{Y}_t$ shall be called the *direction functions* of the development of the system. Let us now denote by ΔX_t and ΔY_t the deviation of the input or output states at the moment t from the values of the direction function of the system. These values shall be called the *norm* of the state of the system and any deviations from the norm shall be called a *disturbance*. Thus, a disturbance is defined as:

$$\Delta X_t = X_t - \hat{X}_t \tag{9.3}$$

and

$$\Delta Y_t = Y_t - \hat{Y}_t . \tag{9.4}$$

The definition of an ergodic process, (9.1) and (9.2), can, therefore, be written in the form

$$\lim_{t\to\infty} \Delta X_t = 0 \tag{9.5}$$

and

$$\lim_{t\to\infty} \Delta Y_t = 0, \tag{9.6}$$

the two expressions being, as we know, equivalent. This means that in an ergodic process any disturbance in the development of the system disappears with time.

The last-mentioned definition of an ergodic process allows us to find the conditions which the couplings of the elements of the system must satisfy in order that the development of the system be an ergodic process. The disturbance ΔX_t or ΔY_t, if small, satisfies formulae (8.9) with the proviso that the functional matrix assumes values corresponding to the norm, i.e. to the values of the direction function, at the moment of the disturbance. Let us denote this moment by $t = z$. We then have

$$\Delta X_{t+\theta} = \left[\frac{\partial R}{\partial X_t}\right]_{X_t = \hat{X}_z} \Delta X_t , \tag{9.7a}$$

$$\Delta X_t = \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}_{z-\tau}} \Delta X_{z-\tau} \tag{9.7b}$$

and

$$\Delta X_t = \int_0^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}_{z-\tau}} \Delta X_{t-\tau}\, d\tau \tag{9.7c}$$

in the place of the formulae (8.9).

We obtain characteristic equations analogous to (8.17) with the proviso that the values of the matrix of the equation correspond to the norm, i.e. to the states $X_t = \hat{X}_z$, or $X_{t-\tau} = \hat{X}_{z-\tau}$. As a result, we find that the condition (both necessary and sufficient) for the development of the system to be an ergodic process is the negative definite character of matrix

$$\left[\frac{\partial R}{\partial X_t}\right]' \left[\frac{\partial R}{\partial X_t}\right]_{X_t=\hat{X}_z} - I, \tag{9.8a}$$

or of appropriate properties of the matrix

$$\sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}}\right]' \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}_{z-\tau}} - I \tag{9.8b}$$

or

$$\int_0^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}}\right]' d\tau \int_0^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}_{z-\tau}} d\tau - I. \tag{9.8c}$$

This gives rise to conclusions similar to those in the case of the stability of a system. Only a system possessing at least one feed-back coupling can develop ergodically; a system with no feed-back coupling is neutral. Furthermore in order that the development of the system be an ergodic process, the feed-back couplings must have certain properties, viz. they must be such that the condition (9.8) be satisfied. Such feed-back couplings shall be called, as in the case of the stability of a system, *compensatory*. Compensatory feed-back couplings reduce the disturbances in the development of a system and, with time, this leads to the disappearance of the disturbances.

Compensatory feed-back couplings in a system which develops ergodically shall also be called the *steering-controls*, or simply,

steerings of the system. The tending of development of the systems to its norm, i.e. to its direction function, in other words: the disappearance of disturbances in the course of development shall be called *self-steering* of the development of the system. Self-regulation of a stable system is a particular case of self-steering of development; it occurs when the direction function is of the form $\hat{\boldsymbol{X}}_t = \text{const.}$, and $\hat{\boldsymbol{Y}}_t = \text{const.}$

The value of the functional matrix which appears in the difference or integral equations (9.7) depends on the value of the direction function at the moment $t = z$, i.e. at the moment when the disturbance $\Delta \boldsymbol{X}_t$ occurs. Therefore, the determinant in the characteristic equation also depends on the time moment $t = z$. In consequence, the roots of the characteristic equation also depend on the moment $t = z$ and are functions of the parameter z; we thus write them in the form $\lambda_j(z)$. The vector function of time $\Delta \boldsymbol{X}_t$ which constitutes the solution of equations (9.7) is, therefore, of the form

$$\Delta \boldsymbol{X}_t = \sum_j \boldsymbol{K}_j(z)\,[\lambda(z)]^t. \tag{9.9}$$

The coefficients $\boldsymbol{K}_j$ also depend on the parameter z and for this reason we write them $\boldsymbol{K}_j(z)$. The function $\Delta \boldsymbol{X}_t$ tends to zero if all $\lambda_j(z) < 1$, which happens when the condition (9.8) is satisfied. Depending on whether $\lambda_j(z)$ is real or complex, the components of the vector function (9.9) are monotonous,[72] or oscillatory. Symbolically the ergodic process can be illustrated by a two-dimensional diagram as on Figs. 10a and 10b.

Figure 10a represents a monotonous, and Fig. 10b an oscillatory disappearance of a disturbance $\Delta \boldsymbol{X}_t$ occurring at the moment $t = z$. In these figures, $\hat{\boldsymbol{X}}_t$ is the direction function of the development of the system and the function $\boldsymbol{X}_t$ represents the actual development of the system in time.

The function $\boldsymbol{X}_t$ representing the development of the system in time is the solution of the vectorial difference or integral equations (9.7). For this reason its shape depends on the input

[27] Here we disregard the short discrete oscillations occurring in discrete process when $\lambda_j(z) < 0$. Cf. footnote 21 on p. 50.

states at certain initial time moments. Similarly, the shape of the function Y_t depends on the output states at certain initial moments. The class of the values of the system's initial states (i.e. of the input or output states) for which the function X_t tends to the direction function $\hat{X}_t$, or Y_t tends to $\hat{Y}_t$, constitutes the *domain of ergodicity* of the development of the system. Disturbances implying states of the system within the domain of ergodicity disappear with time whereas disturbances implying states

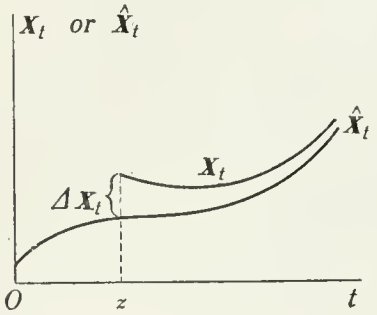

Fig. 10a

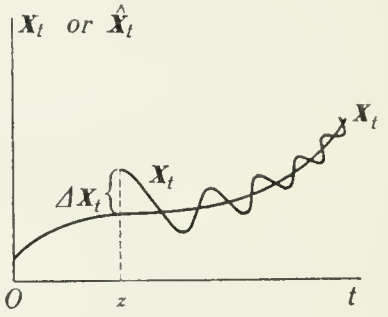

Fig. 10b

of the system outside of that domain do not disappear but exert a lasting influence on the development of the system. The domain of ergodicity may, therefore, also be described as the *domain of transient disturbances* and the set of the system's initial states which do not belong to that domain may be described as the *domain of enduring disturbances.*

If the development of a system is an ergodic process then, by definition, a certain domain in the neighbourhood of the direction function $\hat{X}_t$ or $\hat{Y}_1$ is the domain of ergodicity. This is called the domain of ergodicity "in the small", i.e. for small disturbances. In order to determine the size of the domain of ergodicity it is necessary to investigate the conditions for the domain of ergodicity "in the large", i.e. to find how large disturbances are permissible without the ergodicity of the process of development being upset.

For this purpose we recall that the difference or integral equations (9.7) have been derived for small disturbances ΔX_t.

If the disturbances ΔX_t are large, then instead of (9.7) we should write

$$\Delta X_{t+\theta} = \left[\frac{\partial R}{\partial X_t}\right]_{X_t=\hat{X}_z} \Delta X_t + \Phi(\Delta X_t), \tag{9.10a}$$

$$\Delta X_t = \sum_{\tau=0}^{\theta} \left[\frac{\partial R(\tau)}{\partial X_{t-\tau}}\right]_{X_{t-\tau}=\hat{X}_{z-\tau}} \Delta X_{t-\tau} + \sum_{\tau=0}^{\theta} \Phi(\Delta X_{t-\tau}), \tag{9.10b}$$

$$\Delta X_t = \int_0^{\theta} \left[\frac{\partial R(\tau)}{\partial X_t}\right]_{X_t=\hat{X}_{z-\tau}} \Delta X_{t-\tau}\, d\tau + \int_0^{\theta} \Phi(\Delta X_{t-\tau})\, d\tau. \tag{9.10c}$$

The functions $\Phi(\Delta X_t)$ or $\Phi(\Delta X_{t-\tau})$ on the right-hand sides of these expressions can be transformed into vector functions the arguments of which are the absolute values of the components of the vector ΔX_t or $\Delta X_{t-\tau}$. A function so transformed is denoted by $V(|\Delta X_t|)$ or $V(|\Delta X_{t-\tau}|)$. We thus have

$$\Phi(\Delta X_t) = V(|\Delta X_t|) \quad \text{and} \quad \Phi(\Delta X_{t-\tau}) = V(|\Delta X_{t-\tau}|). \tag{9.11}$$

If the development of the system is an ergodic process, then the domain (i.e. the set of values ΔX_t or $\Delta X_{t-\tau}$) in which

$$\lim_{t\to\infty} V = 0, \tag{9.12}$$

is the domain of ergodicity of the development of the system.[28]

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[28] Instead of defining V as a vector function of the absolute values of the components of the vector ΔX_t or $\Delta X_{t-\tau}$, we can also define V as a function of $(\Delta X_t)^2$ or $(\Delta X_{t-\tau})^2$. The function $V[(\Delta X_t)^2]$ or $V[(\Delta X_{t-\tau})^2]$ so defined is a function of the "Euclidean distance" of the system's state from its norm, i.e. from the value of the direction function at the same moment of time. The use of such a definition is in keeping with the procedure employed by Lyapunov in investigating the stability "in the large" of solutions of differential equations. Such a definition suggests itself when the transformations R_{rs} appearing in the matrix R (cf. (6.18) above) have second derivatives. For then the function Φ is the second term in the Taylor expansion and is equal to

$$\Phi = \frac{1}{2}\left[\frac{\partial^2 R}{\partial X_t^2}\right]_{X_t=\hat{X}_z} (\Delta X_t)^2 \quad \text{or} \quad \Phi = \frac{1}{2}\sum_{\tau=0}^{\theta} \left[\frac{\partial^2 R(\tau)}{\partial X_{t-\tau}^2}\right]_{X_{t-\tau}=\hat{X}_{z-\tau}} (\Delta X_{t-\tau})^2,$$

For in this domain the function $\Phi(\Delta \boldsymbol{X}_t)$ or $\Phi(\Delta \boldsymbol{X}_{t-\tau})$ tends to zero and equations (9.10) converge to equations (9.7) the solution of which, as we have assumed, represents an ergodic proeess.

Besides the question of the domain of ergodicity of the development of the system there is also the question of the duration of ergodieity. For as is seen from (9.9), the function $\Delta \boldsymbol{X}_t$ whieh constitutes the solution of the vectorial difference or integral equations (9.7) depends on the parameter z, that is, on the moment at which the disturbance occurs. In particular, the roots of the charaeteristic equation $\lambda_j(z)$ depend on this parameter. Those roots can be such that $\lambda_j(z) < 1$ for certain values of z and $\lambda_j(z) \geqslant 1$ for other values of z. If the elements of the functional matrix in equations (9.7) are eontinuous funetions of time, which here we shall assume for the sake of simplicity, then the roots of the characteristic equation are continuous funetions of the parameter z. In eonsequence, an interval of time $[t_1, t_2]$ can be defined such that $|\lambda_j(z)| < 1$ for values of z lying within that interval, i.e. for

$$t_1 \leqslant z \leqslant t_2. \tag{9.13}$$

The interval of time so defined shall be ealled the *duration of ergodicity* of the development of the system. If this interval is $-\infty < z < \infty$, we shall say that the process of development of the system is *permanently ergodic*, whereas if the interval is finite, we shall say that the process is *transiently ergodic*. A process can also be transiently ergodie one-sidedly, viz. when $-\infty < z \leqslant t_2$ or $t_1 \leqslant z < \infty$.

A transiently ergodic system can acquire and lose its ergodicity in two ways. One way is when $|\lambda_j(z)|$ are all equal to zero for $z < t_1$ or $z < t_2$, that is, before or after the duration of ergodicity. The system is then transformed from a neutral one into a system developing ergodically, or, conversely, from a system developing ergodieally into a neutral one. Metaphorically,

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or, finally,

$$\Phi = \frac{1}{2} \int_0^{\theta} \left[\frac{\partial^2 R(\tau)}{\partial X_{t-\tau}^2} \right]_{X_{t-\tau} = \hat{X}_{z-\tau}} (\Delta X_{t-\tau})^2 \, d\tau.$$

In view of this, we can write immediately $V = \Phi$.

this can be expressed by saying that a system developing ergodically arises from a "dead" system, or an ergodic system is transformed into a "dead" one. The second way is when, for one or more roots of the characteristic equation, $|\lambda_j(z)| \geqslant 1$ before or after the duration of ergodicity. In the case when $|\lambda_j(z)| = 1$, the system changes from a quasi-stable system into one developing ergodically—or *vice versa*. When $|\lambda_j(z)| > 1$ before or after the duration of ergodicity, every disturbance before or after that time causes the state of the system to move away cumulatively from the direction function of the system's development. A process moving away cumulatively from the system's direction function may be said to be *anti-ergodic*.

The limitation in time of the ergodicity of the process of development of the system is due to the fact that the matrix appearing in the ergodicity condition (9.8) and the transposed matrix appearing there are functions of time. The elements of these matrices are derivatives of the transformations R_{rs} connecting the input of element E_s with the input of element E_r (cf. (6.5) and (7.4) or (7.16) above). These are derivatives of the type

$$\frac{\partial R_{rs}}{\partial x_t^{(r)}} \quad \text{or} \quad \frac{\partial R_{rs}}{\partial x_{t-\tau}^{(r)}}, \quad \text{where} \quad r = 1, 2, \ldots, N; \quad r \neq s.$$

These derivatives depend on the state of the inputs at a given moment, viz. at the moment t or $t - \tau$. In the matrix and in the transposed matrix appearing in (9.8), the state of the inputs has the value of the direction function at the moment $t = z$ or $t - \tau = z - \tau$. In this way, at different moments of time z these derivatives have different values and, therefore, the elements of the matrices appearing in (9.8) have also different values. Consequently, the expression (9.8) may satisfy the conditions for ergodicity at certain moments z while at some other moments it may not do so. It follows that it is possible to have $|\lambda_j(z)| < 1$ at certain moments of time z and $|\lambda_j(z)| \geqslant 1$ at some other moments.

In particular, it may happen that outside of a certain interval of time, the values of the derivatives above the matrix diag-

onal become zero. Then, outside of this interval of time, the system does not have any feed-back couplings and is a neutral system. It may also happen that outside of a certain interval of time the values of the derivatives are such that the matrix (9.8a) becomes positive definite or the matrices (9.8b) and (9.8c) lose the properties required for ergodicity. Then the feed-back couplings in the system cease to be compensatory and all disturbances cause a cumulative process of the system's state moving farther and farther away from the direction function, i.e. the process becomes anti-ergodic.

Attention should be drawn to the speed with which the disturbance in the development disappears in an ergodic process; in other words, the speed with which the system returns to norm after a disturbance. From (9.9) we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta \boldsymbol{X}_t = \sum_j \boldsymbol{K}_j(z)\,[\lambda_j(z)]^t \log|\lambda_j(z)|\,, \tag{9.14}$$

where $\log|\lambda_j(z)|$ is a natural logarithm.[29]

A number of interesting conclusions follow. If, in the neighbourhood of the bounds (or bound) of the duration of ergodicity of the system's development the values $|\lambda_j(z)|$ converge monotonously to unity as z approaches the bound, then $\log|\lambda_j(z)|$ converges monotonously to zero. Hence, as can be seen from (9.14), the speed with which the disturbances disappear also converges to zero. The disturbances in the process of the system's development last longer and longer, the state of the system returns more and more slowly to the norm, i.e. to the state determined by the direction function of the system's development. If this occurs as the upper bound of the duration of ergodicity of the system's development is approached it can be said metaphorically that the system is "growing old"; its self-steering becomes sluggish. If, on the other hand, this happens as the lower

[29] We write $\log|\lambda_j(z)|$ for, if $\lambda_j(z) > 0$, then $\lambda_j(z) = |\lambda_j(z)|$. If, on the other hand, $\lambda_j(z) < 0$, we treat it as a complex number $\lambda_j(z) = \varrho(z)[\cos\omega(z) + \mathrm{i}\sin\omega(z)]$, where $\sin\omega(z) = 0$ and $\cos\omega(z) = 1$. As a result, $\log\lambda_j(z) = \log\varrho(z) = \log|\lambda_j(z)|$. If $\lambda_j(z)$ is a complex number, then $\log\lambda_j(z) = \log\varrho(z) + \log[\cos\omega(z) + \mathrm{i}\sin\omega(z)]$; we then have $\log|\lambda_j(z)| = \log\varrho(z)$.

bound of the duration of ergodicity is approached we can say that the system is increasingly "immature" and that the self-steering of its development has not yet "been broken in".

The same thing happens if, as the bound of the duration of ergodicity is approached by z, $\lambda_j(z)$ converges monotonously to zero. For then $\lambda_j(z) \log |\lambda_j(z)|$ also converges to zero and we have the result just described.[30] Thus, both when the system approaches the neutral state monotonously and when it approaches the quasi-stable state monotonously, the self-steering of its development works more and more slowly.

The "maturing" of the system, i.e. its "breaking in" of its self-steering, and its "ageing" can manifest itself not only in a change of the speed with which disturbances in the development process disappear. It can also be manifest in a change of the domain of ergodicity of the system's development. This domain, as we know, is given by the set of values of the vector $\boldsymbol{X}_t$ or $\boldsymbol{X}_{t-\tau}$ for which (9.12) holds, i.e. by the domain of convergence of the vector function $\boldsymbol{V}(|\Delta \boldsymbol{X}_t|)$ or $\boldsymbol{V}(|\Delta \boldsymbol{X}_{t-\tau}|)$. This function, as a rule, depends on the parameter z, for it contains higher derivatives of the transformations R_{rs} (or $R_{rs}(\tau)$ with respect to $\boldsymbol{x}_t^{(r)} = \hat{\boldsymbol{x}}_z^{(r)}$ $(\boldsymbol{x}_{t+\tau}^{(r)} = \hat{\boldsymbol{x}}_{z-\tau}^{(r)})$ $(r, s = 1, 2, \ldots, N;\ r \neq s)$).[31]

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[30] We write $\lambda(z) \log |\lambda(z)|$ in the form

$$\frac{\log |\lambda(z)|}{[\lambda(z)]^{-1}}.$$

Denoting the derivative by "$'$" and employing de l'Hôpital's rule we have:

$$\lim_{\lambda(z)\to 0} \frac{[\log |\lambda(z)|]'}{\{[\lambda(z)]^{-1}\}'} = \lim_{\lambda(z)\to 0} \frac{1|\lambda(z)|}{-[\lambda(z)]^{-2}} = -\lim_{\lambda(z)\to 0} \lambda(z) = 0.$$

[31] If instead of the function $V(|\Delta X_t|)$ or $V(|\Delta X_{t-\tau}|)$ we use, as in footnote 28 on p. 63, the function of the "Euclidean distance" of the system's state from its norm, we have

$$V[(\Delta X_t)^2] = \frac{1}{2}\left[\frac{\partial^2 R}{\partial X_t^2}\right]_{X_t = \hat{X}_z} (\Delta X_t)^2$$

or

$$V[(\Delta X_{t-\tau})^2 = \frac{1}{2}\sum_{\tau=0}^{\theta}\left[\frac{\partial^2 R(\tau)}{\partial X_{t-\tau}^2}\right]_{X_{t-\tau} = \hat{X}_{z-\tau}} (\Delta X_{t-\tau})^2,$$

The domain of ergodicity of the system's development depends, therefore, on the parameter z, that is, on the moment of time at which the disturbance occurs.

If the ergodicity of the system's development is limited in time it may happen that the domain of ergodicity changes in a certain specific manner. Namely, it may be very small at the beginning of the duration of ergodicity of the system's development, grows with time and later decrease again and converge to zero towards the end of the duration of ergodicity. In such a case there is a change in the resistance of the system's development to larger disturbances. A "very young" system is resistant only to small disturbances, but as it "matures", the system acquires resistance to larger disturbances and finally a system that is "growing old" gradually loses its resistance to larger disturbances and is resistant only to increasingly smaller disturbances until, in the end, it loses all resistance to disturbances.

Thus, the magnitude of the disturbances to which the system is resistant (the domain of ergodicity of its development) as well as the rate at which the system returns to norm after a disturbance may vary in the course of the duration of ergodicity of the system's development.

In an environment producing frequent and large disturbances, many systems lose the ergodic character of their development. These are in particular such systems which had not enough time for "maturing", for "breaking in" their self-steering as well as systems that have small resistance against disturbances because of their being "old" or for some other reason. In this way a natural selection takes place: only those ergodic processes of development (and the corresponding systems) remain which are resistant to a high degree against disturbances, i.e. processes with a large domain of ergodicity and speedy dissapearance of disturbances. Such processes and systems are in a sense "adjusted" to the environment, their self-steering is efficient in the conditions of a given environment.

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or, finally, a similar expression with an integral instead of a sum on the right-hand side. In this case, it is readily seen that vector function V depends on the parameter z.

The existence of ergodic processes, of self-steering of the development of systems explains the phenomena, which through false interpretation has become the basis of metaphysical finalism in conceiving systems as wholes. Such interpretations are frequent, especially in relation to biological phenomena. Such phenomena, as an organism tending to a certain state (homeostasis), regeneration, an organism overcoming of disturbances in its development, constitute the basis of the metaphysical finalistic interpretations of organisms as "wholes". There is the well-known argument based on the experiments by Driesch with embryos of sea urchin. The removal of certain parts (even as much as three-quarters) of an embryo during its early phase of development is compensated during the later stage of development and a whole sea urchin comes into being. From this Driesch inferred that the development of an organism is governed by a non-material guiding force, "entelechy", which imparts to the organism the character of a whole developing according to a certain "plan", according to an immanent purposefulness.[32] Other neo-vitalistic theories in a similar way refer to the "wholistic-purposeful" nature of organisms.

The treatment of the stability and ergodicity of the process of development of a system, which has been presented here and which is based on an analysis of the couplings between the active elements forming the system, shows that metaphysical-finalistic concepts are unnecessary in order to explain the development of systems. The stability of a system and the ergodicity of its process of development are the result of the mode of action of the system's elements and of the couplings of the elements, i.e. of the structure of the system. As can be seen directly from the condition (9.8) for the ergodicity of the system's development (of which stability is a special case) ergodic development processes are the result of the system's law of motion which determines the law of development of the system and all properties

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[32] Cf. Hans Driesch, *Die Philosophie des Organischen*, Leipzig, 1909, Vol. I, p. 59–64.

of that development. As we know, the law of motion of the system reduces to the mode of action of the system's elements and the structure of the system (i.e. the network of couplings)—see above (7.6) and (7.9) as well as (7.17) and (7.18). Ergodic development occurs when the structure of the system contains compensatory feed-back couplings which act as steering controls.

From the law of motion of the system it is possible also to derive such properties of ergodic development processes as the existence of a bounded domain of ergodicity,[33] bounded duration of ergodicity of the development process, changes in the speed with which the system returns to norm after disturbances, and also changes in the system's resistance to disturbances of different magnitudes and the associated "ageing" or "maturing" of systems, their "birth" and "death".

All these properties, characteristic of biological processes also appear in all sorts of physical and chemical, as well as sociological and economic processes.[34] They also appear in automatic devices built by man. The conceptual apparatus created by cybernetics permits them to be explained without resort to the concepts of a non-material being which regulates the course of nature and social development, "vital force", "entelechy", "élan vital", "spirit of the time", or "spirit of a nation", Adam Smith's "invisible hand", etc. At the same time, this explanation does not deny the empirical fact that systems possess a mode of action that cannot be derived from the mere modes of action of their component elements, that they possess their own law of development and that, in certain conditions, the development of systems is an ergodic process in which disturbances

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[33] By too radical an operation it is possible to kill the embryo and then no sea urchin will grow up. The disturbance then went beyond the domain of ergodicity of the system's development. The fact that sometimes monsters are born also indicates that there is a definite domain of ergodicity in the process of embryonic development.

[34] In reference to the "law of motion", directional character of development and "ageing" of socio-economic formations, see Oskar Lange, *Political Economy*, Vol. I., Pergamon Press (Oxford)–PWN Polish Scientific Publishers (Warszawa), 1963.

in the development disappear with time, and, finally, that the ergodicity of the process may be of bounded in duration, that the speed with which disturbances disappear and the magnitude of the disturbances to which the system is resistant may vary, too.

X. THE DIALECTICAL NATURE OF DEVELOPMENT

THE EQUILIBRIUM of a system is expressed by the fact that the input and output states of the elements of the system satisfy the equilibrium equation (8.1). This equation, as we know, has the form of a vector equation and it determines a certain state (or states, if there are more than one) of the inputs and outputs of the elements of the system. The elements, however, are coupled with each other, the output states of some elements are input states of some others. In the case of each element, the state of its inputs determines the state of its outputs; this is the mode of action of the element. Thus it may happen that as a result of the coupling of the elements in a system and the modes of action of the individual elements, it is impossible to realize simultaneously such states of all the inputs and outputs that the equilibrium equation be satisfied. If a certain part of the inputs and outputs assume states satisfying the equilibrium equation then, because of the couplings and the modes of action of the various elements, the remaining inputs and outputs assume states that do not satisfy the equilibrium equation. We then say that from the point of view of equilibrium conditions there is an incompatibility, a contradiction between the states of the inputs and outputs of the individual elements in the system. If the system is stable this contradiction diminishes with time until it finally disappears. On the other hand, in an unstable system this contradiction increases with time, it is this contradiction that produces the cumulative character of the process of development of the system, once the system is thrown out of equilibrium.[35]

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[35] This contradiction between the states of the inputs and outputs of the individual elements in the system has been called "a power of

A system in which the above mentioned contradiction occurs between the states of the inputs and outputs of its individual elements and does not disappear with time, can never be in equilibrium. The states of the inputs and outputs of elements of the system change ceaselessly, in other words, the system is in continual motion. This motion appears as development of the system; it is the result of the contradiction between the states of the inputs and outputs of the individual elements. Given states of the inputs and outputs of some elements in the system cause a change in the input and output states of the remaining elements, this induces a change in the input and output states of the first elements, which, in turn, again causes a change in the states of inputs and outputs of the remaining elements, etc., etc. The motion of the system, its development, is, therefore, a *self-generating dialectical process*, i.e. one in which *contradictions occurring within the system produce its continual motion and development.*

The dialectical character of the motion of the system is expressed by its law of motion which implies the law of development of the system. As we know, the law of motion of a system is expressed mathematically as a vectorial difference or integral equation. This equation—see (7.5) and (7.8) as well as (7.17) and (7.18)—determines the relation between the states of inputs and outputs at one moment of time and the states of these inputs and outputs at a later moment or at later moments. These relations result from the contradictions which prevent the inputs and outputs of all the elements simultaneously to assume states satisfying the equilibrium equation. The input and output states at one moment of time induce certain input

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veto" by W. Ross Ashby. He considers a system consisting of two coupled parts such that when one part is in a state of equilibrium, the states of its inputs and outputs are such that the other part coupled with it cannot be in equilibrium. As a result, the system as a whole can never be in a state of equilibrium since, as Ashby metaphorically puts it, one part of the system applies a veto against the equilibrium of the other part. As a result of this veto the system is in a state of continual motion. Cf. W. Ross Ashby, *op. cit.* pp. 100–101.

and output states at a later moment or at later moments; these, in turn, induce certain input and output states at still later moments, etc. As a result we obtain the process of development of the system in time. The law of development, as we know, is represented mathematically as the solution of a vectorial difference or integral equation expressing the time law of motion of the system. This solution is a vector function of time X_t and Y_t expressing the behaviour in time of the input and output states of all the elements in the system.

As can be seen, the conceptual apparatus of cybernetics—the set of concept peculiar to this science and the appropriate rules of operation—makes possible a precise formulation and solution of two basic problems. One is the problem of "wholes"—of their specific properties and modes of action which cannot be derived from merely the properties and modes of action of the individual elements. The other is the problem of the dialectical nature of development. Examining the problem of "wholes" with the aid of the conceptual apparatus of cybernetics we obtain precise concepts of system and structure. Examining the problem of development, we obtain a precise concept of the dialectical contradiction inherent in a system as well as an explanation as to why this dialectical contradiction is the source of motion and of self-generating development of the system. We also obtain a mathematical formulation of the law of motion and of the law of development of the system. Finally, we obtain a mathematical theory of self-regulation and self-steering of systems, a theory which explains these phenomena without resort to metaphysical constructions. In this way, cybernetics is becoming an important auxilliary instrument for the precise presentation and solution of several basic problems of dialectical materialism.

ERRATA

Page	Instead of	Should be
15_{11}	$(x_1^{(+1)}$	$(x_1^{(q+1)}$
29^{6}	$\overline{X}'$	X'
29^{8}	$\overline{Y}'$	Y'
29^{19}	It suffices in to consider	It suffices to consider
43 (footnote)	$\begin{bmatrix} 0 & \Delta x_{t+\theta}^{(1)} \cdots \Delta x_{t+\theta}^{(1)} \\ \Delta x_{t+\theta}^{(2)} & 0 \quad \cdots \Delta x_{t+\theta}^{(2)} \\ \cdots\cdots\cdots\cdots\cdots \\ \Delta x_{t+\theta}^{(N)} & \Delta x_{t+\theta}^{(N)} \cdots 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \Delta x_{2,t+\theta}^{(1)} \cdots \Delta x_{N,t+\theta}^{(1)} \\ \Delta x_{1,t+\theta}^{(2)} & 0 \quad \cdots \Delta x_{N,t+\theta}^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta x_{1,t+\theta}^{(N)} & \Delta x_{2,t+\theta}^{(N)} \cdots 0 \end{bmatrix}$
43_{3}	(8.9a) or (8.9b)	(8.9b) or (8.9c)
44_{13}	$\sum_{t\to\infty}$	$\lim_{t\to\infty}$
49^{13}	$\Delta X_t = K\varrho^t(\cos\omega t + \mathrm{i}\sin\omega t).$	$\Delta X_t = K\varrho^t \sin\omega t.$
49^{21}	$\dfrac{\partial R_{12}}{\partial x_t^{(1)}} \quad \dfrac{\partial R_{21}}{\partial x_t^{(2)}}$	$\left\vert \dfrac{\partial R_{12}}{\partial x_t^{(1)}} \quad \dfrac{\partial R_{21}}{\partial x_t^{(2)}} \right\vert$
49_{8}	system is	system's
53^{10}	$\Delta X_t = \sum_j K_j(\cos\omega t + \mathrm{i}\sin\omega t).$	$\Delta X_t = \sum_j K_j \sin\omega t.$

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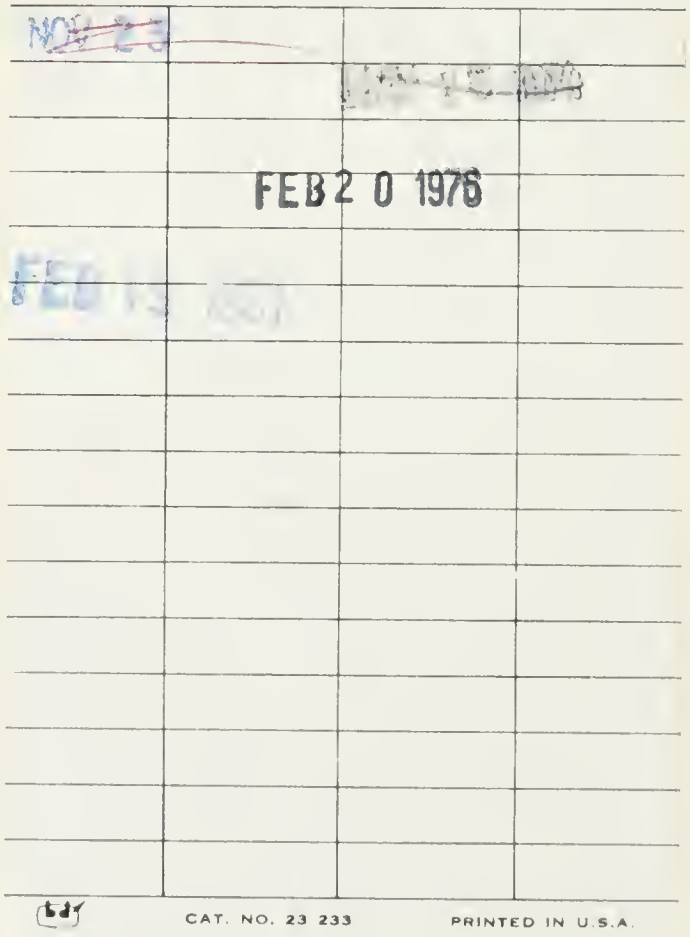

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